

RESONANCES FOR EULER-BERNOULLI OPERATOR ON THE HALF-LINE

ANDREY BADANIN AND EVGENY L. KOROTYAEV

ABSTRACT. We consider resonances for fourth order differential operators on the half-line with compactly supported coefficients. We determine asymptotics of a counting function of resonances in complex discs at large radius, describe the forbidden domain for resonances and obtain trace formulas in terms of resonances. We apply these results to the Euler-Bernoulli operator on the half-line. The coefficients of this operator are positive and constants outside a finite interval. We show that this operator does not have any eigenvalues and resonances iff its coefficients are constants on the whole half-line.

Table of Contents

1. Introduction and main results	1
2. Properties of the free resolvent	8
3. The scattering matrix and the Fredholm determinant	12
4. Proof of the main Theorems and trace formulas in terms of resonances	17
5. Euler-Bernoulli operators and proof of Theorem 1.5	20
6. Resonances for coefficients with jump discontinuity and proof of Theorem 1.3	21

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. There are many results about Schrödinger operators $\mathbf{H} = -\Delta + V(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3$ with compactly supported potentials V on $L^2(\mathbb{R}^3)$, see [Z89], [SZ91] and references therein. In the important physical case the potential $V(|\mathbf{x}|)$ is symmetric and depends only on radius $r = |\mathbf{x}| > 0$. The standard transform $y(\mathbf{x}) \mapsto ry(\mathbf{x})$ and expansion in spherical harmonics give that \mathbf{H} is unitarily equivalent to a direct sum of the Schrödinger operators acting on $L^2(\mathbb{R}_+)$. The first operator from this sum is given by $-\frac{\partial^2}{\partial r^2} + V(r)$. There are a lot of results about the resonances for 1-dimensional case, see [F97], [K04], [S00], [Z87] and references therein.

Now we consider a biharmonic type operator $\mathbf{B} = \frac{1}{\beta(r)}\Delta\alpha(r)\Delta$ on \mathbb{R}^3 , where $\alpha(r), \beta(r)$ are some positive functions depending on the radius r only. The similar operators on \mathbb{R}^2 describe, for example, vibrations of plates with an axisymmetric variable thickness, see [L69]. The separation of variables (similar to the case of Schrödinger operators), show that the operator \mathbf{B} is unitarily equivalent to a direct sum of fourth order operators acting on $L^2(\mathbb{R}_+)$. The first operator from this sum is given by an Euler-Bernoulli operator $\frac{1}{b}\frac{d^2}{dr^2}a\frac{d^2}{dr^2}$ on the half-line with some positive coefficients a, b . Remark that the Euler-Bernoulli operators are related with the problems of vibrations of beams, see [TW59].

The standard unitary Liouville type transformation reduces the Euler-Bernoulli operator into a fourth order operator H on the half-line defined by (1.1) with some coefficients p, q .

Date: April 4, 2017.

1991 *Mathematics Subject Classification.* 34L25 (47E05 47N50).

Key words and phrases. resonances, scattering, fourth order operators.

Thus in order to discuss resonances for Euler-Bernoulli operators we consider a fourth order operators H acting on $L^2(\mathbb{R}_+)$ and given by

$$Hy = H_0y + Vy, \quad H_0 = \partial^4, \quad V = 2\partial p\partial + q, \quad (1.1)$$

with the boundary conditions

$$y(0) = y''(0) = 0, \quad (1.2)$$

where $\partial = \frac{d}{dx}$. We consider the operator $H_0 = \partial^4$ as unperturbed and the operator V is its perturbation. The coefficients p, q are compactly supported and belong to the space \mathcal{H}_0 , where $\mathcal{H}_m = \mathcal{H}_m(\gamma)$, $m = 0, 1, 2, \dots$, is the spaces of functions defined by

$$\mathcal{H}_m = \{f \in L^1_{real}(\mathbb{R}_+) : \text{supp } f \in [0, \gamma], f^{(m)} \in L^1(0, \gamma)\}$$

for some $\gamma > 0$. The boundary conditions (1.2) are taken for reasons of convenience. The operators with other boundary conditions can be considered similarly.

It is well known that the operator H is self-adjoint and is defined on the corresponding form domain, see Sect. 2.4. It has purely absolutely continuous spectrum $[0, \infty)$ plus a finite number of simple real eigenvalues, see Proposition 1.1.

1.2. Schrödinger operators. Before discussion about resonances for the fourth order operators we recall the well known results for Schrödinger operators with compactly supported potentials on the half-line. For $p \in \mathcal{H}_0$ we define a Schrödinger operator h on $L^2(\mathbb{R}_+)$ by

$$h = h_0 - p, \quad \text{where } h_0 f = -f'', \quad f(0) = 0. \quad (1.3)$$

Here the operator h_0 is unperturbed. The operator h has purely absolutely continuous spectrum $[0, \infty)$ plus a finite number of simple negative eigenvalues $e_1 < e_2 < \dots < e_N < 0$. Define an operator

$$y_0(k) = p^{\frac{1}{2}}(h_0 - k^2)^{-1}|p|^{\frac{1}{2}}, \quad p^{\frac{1}{2}} = |p|^{\frac{1}{2}} \text{sign } p, \quad k \in \mathbb{C}_+.$$

Each operator $y_0(k)$ is trace class and is analytic in $k \in \mathbb{C}$. Thus we can define the Fredholm determinant

$$d(k) = \det(1 - y_0(k)), \quad k \in \mathbb{C}. \quad (1.4)$$

It is well known, see, e.g. [F63], that the function $d(k)$ has a finite number of simple zeros $\sqrt{e_1}, \dots, \sqrt{e_N}$ in \mathbb{C}_+ , maybe simple zero at $k = 0$ and an infinite number of zeros (resonances) in \mathbb{C}_- , see Fig. 2 a). We define resonances as zeros of a Fredholm determinant $d(k)$ in $\mathbb{C}_- \cup \{0\}$.

There are a lot of different results about resonances for 1-dimensional Schrödinger operators with compactly supported potentials, see Froese [F97], Hitrik [H99], Korotyaev [K04], Simon [S00], Zworski [Z87] and references therein. Recall the following results:

1) The set of resonances is symmetric with respect to the imaginary axis, since the operator h is self-adjoint.

2) The resonances may have any multiplicity (see [K04]).

3) Let $\gamma = \sup(\text{supp}(p))$ and let $n(r)$ be the number of zeros of $d(k)$ in a disk $|k| < r$. Zworski [Z87] determined the following asymptotics (see also [F97], [K04], [S00])

$$n(r) = \frac{2\gamma r}{\pi} + o(r) \quad \text{as } r \rightarrow \infty.$$

Moreover, for each $\delta > 0$ the number of zeros of $d(k)$ with modulus $\leq r$ lying outside both of the two sectors $|\arg k|, |\arg k - \pi| < \delta$ is $o(r)$ for large r .

4) There are only finitely many resonances in the domain $\{k \in \mathbb{C}_- : |k| > \|p\|e^{\|p\|}e^{-2\gamma \text{Im } k}\}$.

5) Lieb-Thirring type inequalities for resonances were determined in [K12].

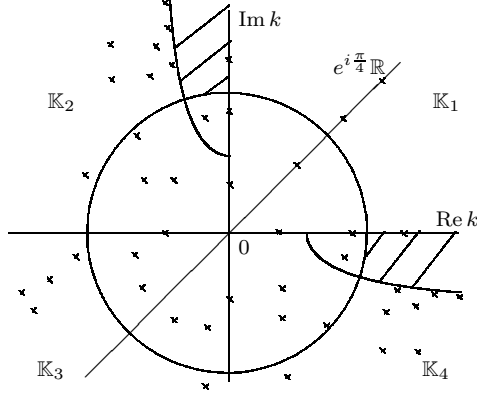


FIGURE 1. The plane of variable k . The function $D(k)$ is real on the line $e^{i\frac{\pi}{4}}\mathbb{R}$. The resonances are marked by crosses. The forbidden domain for the resonances is shaded.

6) Inverse resonance problem was solved (characterization, recovering, plus uniqueness) in terms of resonances for the Schrödinger operator with a compactly supported potential on the real line [K05] and the half-line [K04], see also Zworski [Z02] concerning the uniqueness.

7) Stability estimates for resonances were determined in [K04x], [MSW10].

Thus, the problems of resonances for 1-dimensional Schrödinger operators with compactly supported potentials are well understood.

1.3. Determinant. Instead of the spectral parameter $\lambda \in \mathbb{C}$ we introduce a new variable $k = \lambda^{\frac{1}{4}} \in \mathbb{K}_1$, where \mathbb{K}_1 is the quadrant given by

$$\mathbb{K}_1 = \{k \in \mathbb{C} : \arg k \in (0, \frac{\pi}{2})\}.$$

The mapping $k = \lambda^{\frac{1}{4}} \in \mathbb{C}$ gives a parametrization of the four sheeted Riemann surface Λ of the function $\lambda^{\frac{1}{4}}$ as a complex plane of the variable k . The sheet $\Lambda_j = \mathbb{C} \setminus \mathbb{R}_+$, $j = 1, 2, 3, 4$ of the surface Λ corresponds to the quadrant

$$\mathbb{K}_j = i^{j-1}\mathbb{K}_1, \quad j = 1, 2, 3, 4,$$

of the complex plane of the variable k , see Fig. 1. The first sheet Λ_1 is physical, while the other sheets $\Lambda_2, \Lambda_3, \Lambda_4$ are non-physical.

In order to define the Fredholm determinant we rewrite the perturbation V in the form

$$V = V_1 V_2, \quad V_1 = (\partial |2p|^{\frac{1}{2}}, |q|^{\frac{1}{2}}), \quad V_2 = \begin{pmatrix} (2p)^{\frac{1}{2}} \partial \\ q^{\frac{1}{2}} \end{pmatrix}, \quad p^{\frac{1}{2}} = |p|^{\frac{1}{2}} \text{sign } p. \quad (1.5)$$

We set

$$R_0(k) = (H_0 - k^4)^{-1}, \quad Y_0(k) = V_2 R_0(k) V_1, \quad k \in \mathbb{K}_1. \quad (1.6)$$

In the next proposition we show that each operator $Y_0(k)$, $k \in \mathbb{K}_1$, is trace class and is analytic in the plane without zero. Thus we can define the Fredholm determinant $D(k)$ by

$$D(k) = \det(I + Y_0(k)), \quad k \in \mathbb{C} \setminus \{0\}. \quad (1.7)$$

The function D is analytic in $\mathbb{C} \setminus \{0\}$. Note that $k \in \overline{\mathbb{K}_1} \setminus \{0\}$ is a zero of the determinant D iff $\lambda = k^4 \in \mathbb{R} \setminus \{0\}$ is an eigenvalue of the operator H . We define the resonances as zeros of a Fredholm determinant in $\mathbb{C} \setminus \overline{\mathbb{K}_1}$. We formulate our preliminary results about the determinant.

Proposition 1.1. *Let $p, q \in \mathcal{H}_0$. Then*

i) Each operator $Y_0(k), k \in \mathbb{K}_1$, is trace class and the operator-valued function $kY_0(k)$ is entire.

ii) The Fredholm determinant $D(k)$ is analytic in \mathbb{K}_1 and has an analytic extension into the whole plane without zero such that the function $kD(k)$ is entire. In particular, the operator H has a finite number of eigenvalues. Moreover, $D(k)$ is real on the line $e^{i\frac{\pi}{4}}\mathbb{R}$ and satisfies:

$$D(k) = \overline{D(ik)} \quad \forall k \in \mathbb{C} \setminus \{0\}, \quad (1.8)$$

$$D(k) = 1 - \frac{1+i}{2k} \int_{\mathbb{R}_+} p(x)dx + \frac{O(1)}{k^2} \quad \text{as} \quad |k| \rightarrow \infty, \quad k \in \overline{\mathbb{K}}_1, \quad (1.9)$$

uniformly in $\arg k \in [0, \frac{\pi}{2}]$.

Remark. Due to (1.8) the function $D(k)$ is symmetric with respect to the line $e^{i\frac{\pi}{4}}\mathbb{R}$. Thus it is sufficiently to analyze this function in the half-plane $e^{i\frac{\pi}{4}}\overline{\mathbb{C}}_+$.

Recall that the zeros of the function D in $\mathbb{C} \setminus \overline{\mathbb{K}}_1$ are called *resonances* of H . Let $\mathcal{N}(r)$ be the number of zeros of the function D in the disc $|k| < r$, counted with multiplicity. We present our first main result.

Theorem 1.2. *Let $p, q \in \mathcal{H}_0$ and let $k_* \in \mathbb{K}_2$ be a resonance. Then*

$$|D(k)| \leq Ce^{2\gamma((\operatorname{Re} k)_- + (\operatorname{Im} k)_-)}, \quad \forall k \in \mathbb{C}, \quad |k| \geq 1, \quad (1.10)$$

$$\mathcal{N}(r) \leq \frac{4\gamma r}{\pi}(1 + o(1)) \quad \text{as} \quad r \rightarrow \infty, \quad (1.11)$$

$$|k_*| \leq Ce^{-2\gamma \operatorname{Re} k_*}, \quad (1.12)$$

for some constant $C = C(p, q) > 0$, where $(a)_- = \frac{|a| - a}{2}$.

Remark. 1) Due to (1.12) the domain $\{k \in \mathbb{K}_2 : |k| > Ce^{-2\gamma \operatorname{Re} k}\}$ is forbidden for the resonances in \mathbb{K}_2 and by the symmetry (1.8) of the function $D(k)$, the domain $\{k \in \mathbb{K}_4 : |k| > Ce^{-2\gamma \operatorname{Im} k}\}$ is also forbidden for the resonances in \mathbb{K}_4 , see Fig. 1.

2) Estimate (1.10) is crucial to prove trace formulas in terms of resonances in Theorem 4.2.

3) Consider a specific case when the coefficients p, q satisfy $q = p'' + p^2$. Then H has the form $H = h^2$, where h is the Schrödinger operator, given by (1.3). The determinant D for the operator h^2 satisfies the identity

$$D(k) = d(ik)d(k) \quad \forall k \in \mathbb{C},$$

see [BK16], where $d(k)$ is the determinant for the operator h , given by (1.4). Thus in this case we can describe resonances of H , see Fig. 2 b). For example, due to Zworski [Z87], we obtain the asymptotic distribution of resonances. The resonance of the operator h , and then of h^2 , may have any multiplicity, see [K04]. Moreover, we can determine the forbidden domains for the resonances of the operator $H = h^2$ in all quadrants $\mathbb{K}_2, \mathbb{K}_3, \mathbb{K}_4$ (on all non-physical sheets $\Lambda_2, \Lambda_3, \Lambda_4$).

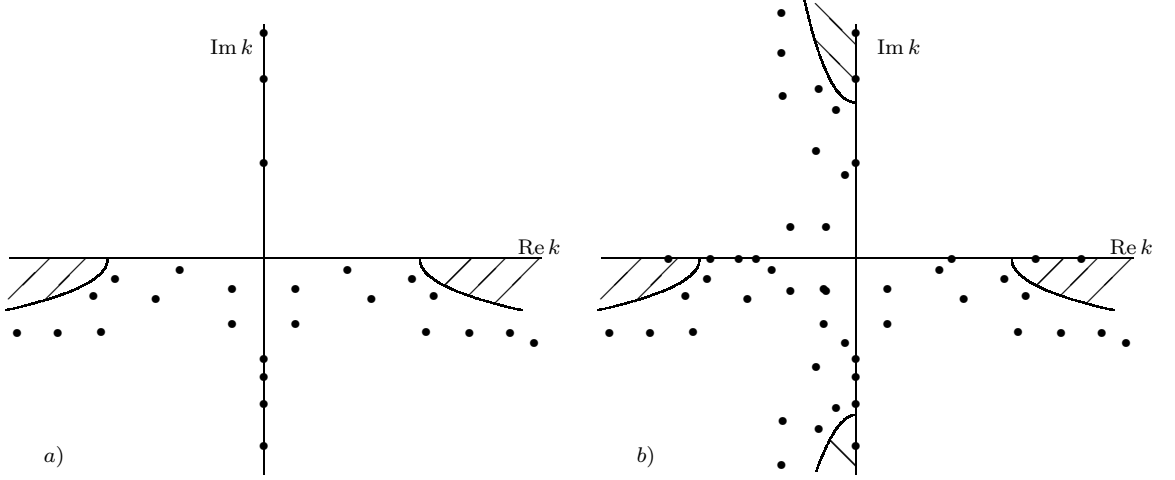


FIGURE 2. a) Resonances for the second order operator h ; b) Resonances for the fourth order operator h^2 . The resonances are marked by circles. The forbidden domains for the resonances are shaded.

1.4. Asymptotics of resonances. In order to determine asymptotics of resonances we assume the stronger conditions for the coefficients p, q given by

$$q \in \mathcal{H}_0, \quad p \in \mathcal{H}_1 \quad \text{and} \quad p_+ := p(\gamma - 0) \neq 0. \quad (1.13)$$

Introduce the numbers $k_{\pm n}^0, n \in \mathbb{N}$ by

$$k_n^0 = \frac{1}{\gamma} \left(i\pi j_n - \log \frac{2\pi n}{\gamma |2p_+|^{\frac{1}{2}}} \right), \quad k_{-n}^0 = ik_n^0 - \frac{\pi}{2\gamma}, \quad n \in \mathbb{N}, \quad (1.14)$$

where $j_n = \begin{cases} n, & \text{if } p_+ > 0 \\ n + \frac{1}{2}, & \text{if } p_+ < 0 \end{cases}$, see Fig. 3.

Theorem 1.3. *Let p, q satisfy (1.13). Then for any $\varepsilon \in (0, \frac{\pi}{2\gamma})$ there exists $\rho > 0$ such that in each disk $\{|k - k_n^0| < \varepsilon\} \subset e^{i\frac{\pi}{4}}\mathbb{C}_+ \cap \{|k| > \rho\}$ there exists exactly one resonance k_n and there are no other resonances in the domain $e^{i\frac{\pi}{4}}\mathbb{C}_+ \cap \{|k| > \rho\}$. These resonances satisfy*

$$k_n = k_n^0 + o(1) \quad \text{as } n \rightarrow \pm\infty. \quad (1.15)$$

In particular, there is a finite number of resonances on $\mathbb{R} \cup i\mathbb{R}$.

Let $\mathcal{N}_j(r), j = 2, 3, 4$ be the number of zeros of the function D in a domain $\mathbb{K}_j \cap \{|k| < r\}$ counted with multiplicity, where $\mathbb{K}_j = i^{j-1}\mathbb{K}_1$ and let $r \rightarrow \infty$. Then we have

$$\begin{aligned} \mathcal{N}_2(r) &= \mathcal{N}_4(r) = \frac{\gamma r}{\pi} (1 + o(1)), \\ \mathcal{N}_3(r) &= \frac{2\gamma r}{\pi} (1 + o(1)). \end{aligned} \quad (1.16)$$

Remark. 1) Due to the identity (1.8), for each resonance in the domain $e^{i\frac{\pi}{4}}\mathbb{C}_+$ there exists the symmetric resonance in $e^{i\frac{\pi}{4}}\mathbb{C}_-$ with the same multiplicity. Thus Theorem 1.3 describes all resonances outside the large disc.

2) Roughly speaking the proof of Theorem 1.3 is based on the fact that the Born approximation is the “main term” of the scattering amplitude at large k .

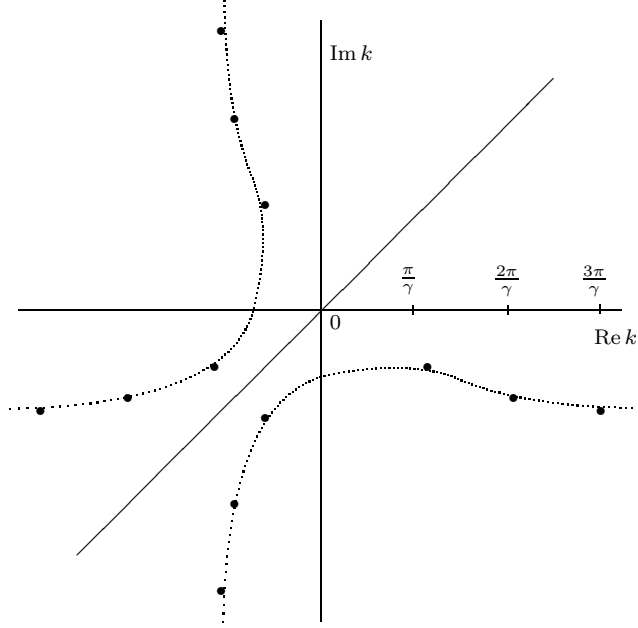


FIGURE 3. Asymptotics of resonances for step coefficients

3) From (1.16) we obtain $\mathcal{N}_3(r) = 2\mathcal{N}_2(r)(1+o(1))$. This asymptotics shows the distribution of the resonances on the Riemann surface Λ of the function $\lambda^{\frac{1}{4}}$: the number of resonances in the large disc on the sheet Λ_3 is in two times more than on the sheet Λ_2 (and Λ_4). Note that the corresponding question for a second order operator has no meaning, since the Riemann surface for this case has only one non-physical sheet.

1.5. Euler-Bernoulli operators. We consider an Euler-Bernoulli operator $\mathcal{E} \geq 0$ acting on $L^2(\mathbb{R}_+, b(x)dx)$ and given by

$$\mathcal{E}u = \frac{1}{b}(au'')'',$$

with the boundary conditions

$$u = 0, \quad \text{and} \quad u'' + \frac{a'}{5a}u' = 0 \quad \text{at} \quad x = 0. \quad (1.17)$$

We assume that the coefficients a, b are positive on the unit interval $[0, 1]$ and $a = b = 1$ outside this interval and satisfy

$$a - 1, b - 1 \in \mathcal{H}_4, \quad \left(\frac{3a'}{a} + \frac{5b'}{b} \right) \Big|_{x=0} = 0. \quad (1.18)$$

The Euler-Bernoulli operator describes the relationship between the thin beam's deflection and the applied load, a is the rigidity and b is the density of the beam, see, e.g., [TW59]. The boundary conditions (1.17) mean that the end of the beam is restrained by some special rotational spring device.

The standard Liouville type transformation (see Sect 5) yields that the operator $\mathcal{E} \geq 0$ with the boundary conditions (1.17) is unitarily equivalent to a fourth order operator $H \geq 0$ with the boundary conditions (1.2) and with specific coefficients $p, q \in \mathcal{H}_0$. Using this transformation we can define the determinant for the Euler-Bernoulli operator \mathcal{E} . The resonances for the operator \mathcal{E} coincide with the resonances for the operator H . Thus all results for the

resonances of the operator H can be carried over the operator \mathcal{E} . In particular, we have the following corollary of Theorem 1.2.

Corollary 1.4. *Let the coefficients a, b satisfy the conditions (1.18). Then the determinant $D(k)$, the counting function $\mathcal{N}(r)$ and the resonances for the Euler-Bernoulli operator \mathcal{E} satisfy the estimates (1.10)–(1.12), where*

$$\gamma = \int_0^1 \left(\frac{b(x)}{a(x)} \right)^{\frac{1}{4}} dx. \quad (1.19)$$

Remark. There is an interesting problem to study resonances for the Euler-Bernoulli operator under the condition $a - 1, b - 1 \in \mathcal{H}_1$.

We recall the Borg type uniqueness result for the Euler-Bernoulli operator on a finite interval. We consider the operator $\mathbf{E} = \frac{1}{b}(au'')''$ on the interval $[0, 1]$ with the boundary conditions

$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

We assume that the coefficients a, b are positive and satisfy:

$$a, a''''', b, b'''' \in L^1(0, 1), \quad \int_0^1 \left(\frac{b}{a} \right)^{\frac{1}{4}} dx = 1, \quad a'(0) = a'(1) = b'(0) = b'(1) = 0.$$

It was proved in [BK15] that the eigenvalues of the operator \mathbf{E} are $(\pi n)^4$ for all $n \geq 1$ iff $a = b = 1$.

Now we consider a Borg type result for resonance scattering of the Euler-Bernoulli operator \mathcal{E} on the half-line.

Theorem 1.5. *Let the coefficients a, b satisfy the conditions (1.18). Then the operator \mathcal{E} does not have any eigenvalues and resonances iff $a = b = 1$ on the whole half line.*

Remark. Assume that $a'(0) = b'(0) = 0$. Then the boundary conditions (1.17) take the form of the conditions for a pinned beam: $u(0) = u''(0) = 0$. Moreover, in this case the last condition in (1.18) also holds true.

1.6. Historical review. A lot of papers are devoted to the direct and inverse spectral problems for fourth order operators on the line: Aktosun and Papanicolaou [AP08], Butler [B68], Beals [B85], Iwasaki [I88], [I88x], Hoppe, Laptev and Östensson [HLO06], Laptev, Shterenberg, Sukhanov and Östensson [LSSO06]. Moreover, there is a paper [B85] and even a book [BDT88] about scattering for 1-dimensional higher order operators. However, even the inverse scattering problems for fourth order operators on the line (or half-line) are not solved and there are no results about resonances.

Eigenvalue asymptotics for fourth order operators and for the Euler-Bernoulli operators on the finite interval were determined by Badanin and Korotyaev [BK15]. Eigenvalue asymptotics for fourth order operators on the circle were the subject of our paper [BK14], see also Mikhailets and Molyboga [MM12] for the case of distribution coefficients.

We recall that resonances, from a physicists point of view, were first studied by Gamov [Ga]. Since then, properties of resonances have been the object of intense study and we refer to [SZ91] for the mathematical approach in the multi-dimensional case and references given therein. We discuss the resonances for one-dimensional systems and higher order operators. The properties of higher order operators are very different from the properties of second order systems. In particular, all fundamental solutions of a second order system are bounded

on the real axis, while a higher order equation has always exponentially growing solutions. Moreover, the Riemann surface for higher order operators is more complicated, than the Riemann surface for Schrödinger operator with matrix-valued potentials. Thus the study of higher order operators require substantial modifications of methods used in the study of matrix second order operators, see, e.g., [I88], [I88x]. Nedelec [N07] considered the resonances for Schrödinger operator with matrix-valued compactly supported potentials on the line. The resonance scattering for third order operators on the line was considered in [K16]. It is a first paper on the resonances for higher order operators. Resonance are defined as zeros of a Fredholm determinant on a non-physical sheet of three sheeted Riemann surface. Here upper bounds of the number of resonances in complex discs at large radius and the trace formula in terms of resonances only are obtained. The asymptotics of counting function for resonances is still open problem, since the standard Born approximation is not a main term in the high energy asymptotics for the scattering amplitude, as we have for second and fourth order operators. In the present paper we use different methods from [K16]. Note that the situation for a fourth order operator is simpler, than for a third one, because the Born term helps us to obtain asymptotics of counting function for resonances at large radius.

2. PROPERTIES OF THE FREE RESOLVENT

2.1. The well-known facts. By \mathcal{B} we denote the class of bounded operators. Let \mathcal{B}_1 and \mathcal{B}_2 be the trace and the Hilbert-Schmidt class equipped with the norm $\|\cdot\|_{\mathcal{B}_1}$ and $\|\cdot\|_{\mathcal{B}_2}$ correspondingly. We recall some well known facts. Let $A, B \in \mathcal{B}$ and $AB, BA, X \in \mathcal{B}_1$. Then

$$\mathrm{Tr} AB = \mathrm{Tr} BA, \quad (2.1)$$

$$\det(I + AB) = \det(I + BA), \quad (2.2)$$

$$\text{the mapping } X \rightarrow \det(I + X) \text{ is continuous on } \mathcal{B}_1, \quad (2.3)$$

$$|\det(I + X)| \leq e^{\|X\|_{\mathcal{B}_1}}, \quad (2.4)$$

for all $X \in \mathcal{B}_1$, see e.g., Sect. 3. in the book [S05]. Let the operator-valued function $X : \mathcal{D} \rightarrow \mathcal{B}_1$ be analytic for some domain $\mathcal{D} \subset \mathbb{C}$ and $(I + X(z))^{-1} \in \mathcal{B}$ for any $z \in \mathcal{D}$. Then for the function $F(z) = \det(I + X(z))$ we have

$$F'(z) = F(z) \mathrm{Tr}(I + X(z))^{-1} X'(z), \quad z \in \mathcal{D}. \quad (2.5)$$

Introduce the space $L^p(\mathbb{R}_+)$ equipped with the norm

$$\|f\|_p = \left(\int_{\mathbb{R}_+} |f(x)|^p dx \right)^{\frac{1}{p}} \geq 0, \quad \text{and let} \quad \|f\| = \|f\|_2.$$

2.2. Schrödinger operators. Let the Schrödinger operator h and the unperturbed operator h_0 be defined by (1.3).

- The free resolvent $r_0(k) = (h_0 - k^2)^{-1}$, $k \in \mathbb{C}_+$ is an integral operator having the kernel $r_0(x, y, k)$, $x, y \in \mathbb{R}_+$ given by

$$r_0(x, y, k) = \frac{i}{2k} \left(e^{ik|x-y|} - e^{ik(x+y)} \right), \quad k \in \mathbb{C}_+. \quad (2.6)$$

- Define the operator-valued function $g(k) = \alpha r_0(k) \beta$, $k \in \mathbb{C}_+$, for some $\alpha^2, \beta^2 \in \mathcal{H}_0$. For each $k \in \mathbb{C}_+$ the operator $g(k) \in \mathcal{B}_j$, $j = 1, 2$ and the mapping

$$g(k) : \mathbb{C}_+ \rightarrow \mathcal{B}_j \quad (2.7)$$

is analytic and it has an analytic extension into whole complex plane without zero. Thus the operator-valued function $g : \mathbb{C} \setminus \{0\} \rightarrow \mathcal{B}_1$ is analytic. Moreover, we have the following estimate

$$\|g(k)\|_{\mathcal{B}_2} \leq \frac{\|\alpha\|\|\beta\|}{|k|}, \quad \forall \quad k \in \overline{\mathbb{C}}_+ \setminus \{0\}.$$

Define the Fourier transformation $\Phi_s : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$ by

$$\tilde{f}(\xi) = (\Phi_s f)(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \xi x dx, \quad \xi \in \mathbb{R}_+. \quad (2.8)$$

Then $r_0(k) = \Phi_s^* \eta_{-k} \eta_k \Phi_s$, where $\eta_k(\xi)$ is the multiplication by $(\xi - k)^{-1}$ and we have

$$\begin{aligned} \|g(k)\|_{\mathcal{B}_1} &= \|\alpha(h_0 - k^2)^{-1}\beta\|_{\mathcal{B}_1} \leq \|\alpha\Phi_s^* \eta_{-k}\|_{\mathcal{B}_2} \|\eta_k \Phi_s \beta\|_{\mathcal{B}_2} \\ &\leq \frac{2\|\alpha\|\|\beta\|}{\pi} \|\eta_{-k}\| \|\eta_k\| \leq \frac{2\|\alpha\|\|\beta\|}{\operatorname{Im} k}, \quad \operatorname{Im} k > 0, \end{aligned} \quad (2.9)$$

since $\int_0^\infty |\xi \pm k|^{-2} d\xi \leq \frac{\pi}{\operatorname{Im} k}$.

• The Schrödinger equation $-y'' - py = k^2 y$, $k \in \mathbb{C} \setminus \{0\}$, $p \in \mathcal{H}_0$, has a unique Jost solution $f_+(x, k)$ satisfying the condition $f_+(x, k) = e^{ikx}$, $x > \gamma$. The Jost function $f_+(0, k)$ is entire and satisfies

$$f_+(0, k) = 1 + o(1) \quad \text{as } |k| \rightarrow \infty, \quad (2.10)$$

uniformly in $\arg k \in [0, \pi]$. The function $f_+(0, k)$ has a finite number of simple zeros $i\kappa_1, \dots, i\kappa_N$ in \mathbb{C}_+ and the functions $f_+(x, i\kappa_n)$, $n = 1, \dots, N$, are eigenfunctions of h corresponding to the eigenvalues $-\kappa_n^2$. Moreover, $f_+(0, k)$ has an infinite number of zeros (resonances) in \mathbb{C}_- . There are not any zeros on the real axis with only exception $k = 0$, where the function $f_+(0, k)$ may have a simple zero.

The determinant $d(k)$, given by the definition (1.4), satisfies the identity $d(k) = f_+(0, k)$ for all $k \in \mathbb{C}$. The scattering matrix $s(k)$ for the pair h, h_0 has the form

$$s(k) = \frac{d(-k)}{d(k)}, \quad k > 0. \quad (2.11)$$

Using (2.10) we can define the function $\log d(k)$ for large $|k|$, $k \in \mathbb{C}_+$, by $\log d(k) = o(1)$ as $\operatorname{Im} k \rightarrow \infty$. It satisfies

$$i \log d(k) = \frac{1}{2k} \left(\int_{\mathbb{R}_+} p(x) dx + o(1) \right) \quad \text{as } \operatorname{Im} k \rightarrow +\infty.$$

2.3. The free resolvent. The free resolvent $R_0(k) = (H_0 - k^4)^{-1}$, $k \in \mathbb{K}_1$ has a representation in terms of the resolvent r_0 for h_0 :

$$R_0(k) = \frac{r_0(k) - r_0(ik)}{2k^2}, \quad \partial R_0(k) \partial = -\frac{r_0(k) + r_0(ik)}{2}. \quad (2.12)$$

Then, by (2.6), the operator $R_0(k)$ is an integral operator with the kernel $R_0(x, y, k)$ given by

$$R_0(x, y, k) = \frac{i}{4k^3} \left(e^{ik|x-y|} - e^{ik(x+y)} + ie^{-k|x-y|} - ie^{-k(x+y)} \right), \quad x, y > 0. \quad (2.13)$$

For each $x, y \in \mathbb{R}_+$ the function $R_0(x, y, \cdot)$ is analytic in $\mathbb{C} \setminus \{0\}$ and has a simple pole at $k = 0$:

$$R_0(x, y, k) = \frac{(1+i)}{4k} xy + \frac{(i-1)}{24} ((x+y)^3 - |x-y|^3) + O(k) \quad (2.14)$$

as $k \in \mathbb{C}, |k| \rightarrow 0$ uniformly in bounded $x, y \in \mathbb{R}_+$. Let $\alpha^2, \beta^2 \in \mathcal{H}_0$. The last asymptotics shows that each function $\alpha(x)(R_0(x, y, k) - \frac{(1+i)}{4k}xy)\beta(y), x, y \in \mathbb{R}_+$ is entire in k and the function $\frac{(1+i)}{4k}x\alpha(x)y\beta(y)$ is a kernel of the rank one operator.

Define the operator-valued function $G(k) = \alpha \partial R_0(k) \beta, k \in \mathbb{K}_1$, where $\alpha^2, \beta^2 \in \mathcal{H}_0$. The identity (2.12) yields that for each $k \in \mathbb{K}_1$ the operator $G(k) \in \mathcal{B}_j, j = 1, 2$ and the mappings

$$G(k) : \mathbb{K}_1 \rightarrow \mathcal{B}_j \quad (2.15)$$

is analytic and it has an analytic extension into whole complex plane without zero. Moreover, from (2.13) we have the following estimate

$$\|G(k)\|_{\mathcal{B}_2} \leq \frac{\|\alpha\| \|\beta\|}{|k|^2}, \quad k \in \overline{\mathbb{K}_1} \setminus \{0\}.$$

Moreover, we obtain $\partial R_0(k) = \Phi_s^* \rho_{ik} \rho_k \Phi_s$, where $\rho_k(\xi)$ is the multiplication by $(\xi^2 - k^2)^{-1} \xi^{\frac{1}{2}}$ and we have

$$\|G(k)\|_{\mathcal{B}_1} \leq \|\alpha \Phi_s^* \rho_{ik}\|_{\mathcal{B}_2} \|\rho_k \Phi_s \beta\|_{\mathcal{B}_2} \leq \frac{2\|\alpha\| \|\beta\|}{\pi} \|\rho_{ik}\| \|\rho_k\| \leq \frac{\|\alpha\| \|\beta\|}{2 \operatorname{Re} k \operatorname{Im} k}, \quad (2.16)$$

$k \in \mathbb{K}_1$, since $\int_0^\infty \xi |\xi^2 \pm k^2|^{-2} d\xi \leq \frac{\pi}{4 \operatorname{Re} k \operatorname{Im} k}$.

2.4. Resolvent estimates. The operator H_0 is self-adjoint on the form domain $\mathcal{D}(H_0) = \{y, y'' \in L^2(\mathbb{R}_+), y(0) = 0\}$. The quadratic form (Vy, y) is defined by

$$(Vy, y) = -(2py', y') + (qy, y), \quad y \in \mathcal{D}(H_0),$$

where (\cdot, \cdot) is the scalar product in $L^2(\mathbb{R}_+)$. Then the standard arguments give

$$|(Vy, y)| \leq \frac{1}{2} \|y''\|^2 + C \|y\|^2 \quad \forall y \in \mathcal{D}(H_0), \quad (2.17)$$

see e.g., [K03], for some constant $C > 0$. Then the KLMN theorem (see [RS75, Th X.17]) yields that there exists a unique self-adjoint operator $H = H_0 + V$ with the form domain, which coincide with $\mathcal{D}(H_0)$, and

$$(Hy, y_1) = (H_0 y, y_1) + (Vy, y_1) \quad \forall y, y_1 \in \mathcal{D}(H_0), \quad (2.18)$$

In order to obtain resolvent estimates we need to discuss the operator Y_0 . The definitions (1.5), (1.6) imply

$$Y_0 = \begin{pmatrix} (2p)^{\frac{1}{2}} \partial R_0 \partial |2p|^{\frac{1}{2}} & (2p)^{\frac{1}{2}} \partial R_0 |q|^{\frac{1}{2}} \\ q^{\frac{1}{2}} R_0 \partial |2p|^{\frac{1}{2}} & q^{\frac{1}{2}} R_0 |q|^{\frac{1}{2}} \end{pmatrix}, \quad (2.19)$$

where $p^{\frac{1}{2}} = |p|^{\frac{1}{2}} \operatorname{sign} p$. We introduce the operator-valued function Y by

$$Y(k) = V_2 R(k) V_1, \quad k \in \mathbb{K}_1. \quad (2.20)$$

This operator satisfies the standard equation:

$$(I - Y(k))(I + Y_0(k)) = I \quad \forall k \in \mathbb{K}_1 \setminus \sigma_d, \quad (2.21)$$

where σ_d is the set of zeros of the function D in $\overline{\mathbb{K}_1}$.

Lemma 2.1. *Let $p, q \in \mathcal{H}_0$. Then*

i) The operator $Y_0(k) \in \mathcal{B}_j, j = 1, 2$ for each $k \in \mathbb{K}_1$, the operator-valued function $Y_0 : \mathbb{K}_1 \rightarrow \mathcal{B}_j$ is analytic and has an analytic extension into the whole complex plane without zero. The operator-valued function $kY_0(k)$ is entire. Moreover, Y_0 satisfies

$$\|Y_0(k)\|_{\mathcal{B}_2} \leq \frac{C}{|k|}, \quad k \in \overline{\mathbb{K}_1}, \quad (2.22)$$

$$\|Y_0(k)\|_{\mathcal{B}_1} \leq 2(2\|p\|_1 + \|q\|_1) \left(\frac{1}{\operatorname{Re} k} + \frac{1}{\operatorname{Im} k} \right) \left(1 + \frac{1}{|k|} \right)^2, \quad k \in \mathbb{K}_1 \quad (2.23)$$

for some constant $C = C(p, q) > 0$.

ii) Each $Y(k) \in \mathcal{B}_j, j = 1, 2, k \in \mathbb{K}_1 \setminus \sigma_d$ and the operator-valued function $Y : \mathbb{K}_1 \setminus \sigma_d \rightarrow \mathcal{B}_j$ is analytic and has a meromorphic extension into the whole complex plane. Moreover, Y satisfies

$$\|Y(k)\|_{\mathcal{B}_2} \leq \frac{O(1)}{|k|}, \quad (2.24)$$

$$\|Y(k) - Y_0(k)\|_{\mathcal{B}_2} \leq \frac{O(1)}{|k|^2}, \quad (2.25)$$

as $k \in \overline{\mathbb{K}_1}, |k| \rightarrow \infty$.

Proof. i) The definition (1.6) and the identity (2.13) yield (2.22). Substituting the identities (2.12) into (2.19) and using the facts about the mappings g, G in (2.7), (2.15) we deduce that the operator-valued function $Y_0 : \mathbb{K}_1 \rightarrow \mathcal{B}_1$ is analytic and has an analytic extension into the whole complex plane without zero. The asymptotics (2.14) shows that the operator-valued function $kY_0(k)$ is entire.

Using the estimates (2.9) we obtain for $\operatorname{Im} k > 0$:

$$\begin{aligned} \|p^{\frac{1}{2}}r_0(k)|p|^{\frac{1}{2}}\|_{\mathcal{B}_1} &\leq \frac{4\|p\|_1}{\operatorname{Im} k}, & \|q^{\frac{1}{2}}r_0(k)|q|^{\frac{1}{2}}\|_{\mathcal{B}_1} &\leq \frac{2\|q\|_1}{\operatorname{Im} k}, \\ \|p^{\frac{1}{2}}r_0(k)|q|^{\frac{1}{2}}\|_{\mathcal{B}_1} &\leq \frac{2\sqrt{2}(\|p\|_1\|q\|_1)^{\frac{1}{2}}}{\operatorname{Im} k}, \end{aligned}$$

and the similar estimates with $r_0(ik)$. These estimates and the relations (2.16), (2.19) give

$$\|Y_0(k)\|_{\mathcal{B}_1} \leq (\|2p\|_1^{\frac{1}{2}} + \|q\|_1^{\frac{1}{2}})^2 \left(\frac{1}{\operatorname{Re} k} + \frac{1}{\operatorname{Im} k} \right) \left(1 + \frac{1}{|k|} \right)^2,$$

which yields (2.23).

ii) For $k \in \mathbb{K}_1 \setminus \sigma_d$ identity (2.21) gives

$$Y(k) = I - (I + Y_0(k))^{-1} = Y_0(k)(I + Y_0(k))^{-1} \in \mathcal{B}_j, \quad j = 1, 2, \quad (2.26)$$

and, since $Y_0(k)$ is analytic in \mathbb{K}_1 , $Y(k)$ is analytic in $\mathbb{K}_1 \setminus \sigma_d$. Due to the analytic Fredholm theorem, see [RS72, Th VI.14], the function $Y(k)$ has a meromorphic extension into the whole complex plane. The estimate (2.22) implies the asymptotics (2.24). Moreover,

$$Y(k) - Y_0(k) = Y_0(k)((I + Y_0(k))^{-1} - I),$$

which yields (2.25). ■

3. THE SCATTERING MATRIX AND THE DETERMINANT

3.1. The spectral representation for H_0 . The unitary transformation (2.8) carries over H_0 into multiplication by k^4 in $L^2(\mathbb{R}_+, dk)$:

$$(\Phi_s H_0 \Phi_s^* \tilde{f})(k) = k^4 \tilde{f}(k), \quad k > 0, \quad \text{where} \quad \tilde{f}(k) = (\Phi_s f)(k).$$

Define a functional $\psi_1(k) : L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+) \rightarrow \mathbb{C}$ by

$$\psi_1(k)f = (\Phi_s V_1 f)(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(-k|2p(x)|^{\frac{1}{2}} \cos kx, |q(x)|^{\frac{1}{2}} \sin kx \right) f(x) dx, \quad k > 0, \quad (3.1)$$

$f \in L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$. Define an operator $\psi_2(k) : \mathbb{C} \rightarrow L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$, $k > 0$, by

$$\psi_2(k)c = V_2 \sqrt{\frac{2}{\pi}} \sin(kx) c = \sqrt{\frac{2}{\pi}} \begin{pmatrix} k(2p(x))^{\frac{1}{2}} \cos kx \\ q(x)^{\frac{1}{2}} \sin kx \end{pmatrix} c, \quad c \in \mathbb{C}. \quad (3.2)$$

The operator-valued function $\psi_j(k)$, $k \in \mathbb{R}_+$, $j = 1, 2$ have analytic extensions from \mathbb{R}_+ into the whole complex plane. Then we can introduce the operators $\Psi_1(k)$ and $\Psi_2(k)$ by

$$\Psi_1(k) = \begin{pmatrix} \psi_1(ik) \\ \psi_1(k) \end{pmatrix}, \quad \Psi_2(k) = \begin{pmatrix} i\psi_2(ik), \psi_2(k) \end{pmatrix}, \quad k \in \mathbb{C}. \quad (3.3)$$

Lemma 3.1. *Let $p, q \in \mathcal{H}_0$. Then the operator-valued functions $\psi_j(k)$, $j = 1, 2$, are entire and satisfy*

$$\|\psi_j(k)\| \leq \sqrt{\frac{2}{\pi}} \left(|k|^2 \|2p\|_1 + \|q\|_1 \right)^{\frac{1}{2}} e^{\gamma |\operatorname{Im} k|}. \quad (3.4)$$

Proof. The functional $\psi_1(k)$, $k \in \mathbb{C}$ satisfies

$$\begin{aligned} \psi_1(k)f &= \sqrt{\frac{2}{\pi}} \int_0^\gamma \left(-k|2p(x)|^{\frac{1}{2}} \cos kx f_1(x) + |q(x)|^{\frac{1}{2}} \sin kx f_2(x) \right) dx, \\ |\psi_1(k)f|^2 &\leq \frac{2e^{2\gamma |\operatorname{Im} k|}}{\pi} \left(2|k|^2 \|p\|_1 \|f_1\|^2 + \|q\|_1 \|f_2\|^2 \right), \quad f = (f_1, f_2) \in L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+), \end{aligned}$$

which yields (3.4) for ψ_1 . Similarly,

$$\|\psi_2(k)c\|^2 = \frac{2|c|^2}{\pi} \int_0^\gamma \left(|k|^2 |2p(x)| \cos^2 kx + |q(x)| \sin^2 kx \right) dx, \quad c \in \mathbb{C},$$

which yields (3.4) for ψ_2 . ■

For $k \in \mathbb{C} \setminus \{0\}$ we introduce finite rank operators $P_1(k), P_2(k)$ on $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$ by

$$\begin{aligned} P_1(k) &= c_k \psi_2(k) \psi_1(k), \quad c_k = \frac{\pi}{i2k^3}, \\ P_2(k) &= P_1(ik) + P_1(k) = c_k \Psi_2(k) \Psi_1(k) = c_k (i\psi_2(ik) \psi_1(ik) + \psi_2(k) \psi_1(k)). \end{aligned} \quad (3.5)$$

Below we need the following simple identities.

Lemma 3.2. *Let $p, q \in \mathcal{H}_0$ and let $k \in \mathbb{C} \setminus \{0\}$. Then the operators $P_1(k), P_2(k)$ satisfy*

$$P_1(k) = Y_0(ik) - Y_0(k), \quad (3.6)$$

$$P_2(k) = Y_0(-k) - Y_0(k). \quad (3.7)$$

Proof. The identity (2.12) implies

$$R_0(k) - R_0(ik) = \frac{r_0(k) - r_0(-k)}{2k^2}. \quad (3.8)$$

The identity (2.6) shows that the kernel of the integral operator $r_0(k) - r_0(-k)$ has the form

$$r_0(x, y, k) - r_0(x, y, -k) = \frac{i}{k} (\cos k(x - y) - \cos k(x + y)) = \frac{2i}{k} \sin kx \sin ky. \quad (3.9)$$

The definitions (3.1), (3.2) imply

$$(\psi_2(k)\psi_1(k)f)(x) = \frac{2}{\pi} V_2 \left(\sin kx \int_0^\infty \sin ky (V_1 f)(y) dy \right).$$

The identities (3.8), (3.9) and the definition (1.6) give

$$Y_0(ik) - Y_0(k) = V_2(R_0(ik) - R_0(k))V_1 = c_k \psi_2(k)\psi_1(k),$$

which yields the identity (3.6). The identities

$$Y_0(-k) - Y_0(k) = Y_0(-k) - Y_0(ik) + Y_0(ik) - Y_0(k) = P_1(ik) + P_1(k)$$

give (3.7). ■

3.2. The scattering matrix. It is well known that the wave operators $W_\pm = W_\pm(H, H_0)$ for the pair H_0, H , given by

$$W_\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} \quad \text{as} \quad t \rightarrow \pm\infty,$$

exist and are complete, i.e., $\text{Ran } W_\pm = \mathcal{H}_{ac}(H)$. The scattering operator $S = W_+^* W_-$ is unitary. The operators H_0 and S commute and thus are simultaneously diagonalizable:

$$L^2(\mathbb{R}_+) = \int_{\mathbb{R}_+}^\oplus \mathcal{H}_\lambda d\lambda, \quad H_0 = \int_{\mathbb{R}_+}^\oplus \lambda I_\lambda d\lambda, \quad S = \int_{\mathbb{R}_+}^\oplus S(\lambda^{\frac{1}{4}}) d\lambda; \quad (3.10)$$

here I_λ is the identity in the fiber space $\mathcal{H}_\lambda = \mathbb{C}$ and $S(k), k = \lambda^{\frac{1}{4}} > 0$ is the scattering matrix (which is a scalar function in our case) for the pair H_0, H . The stationary representation for the scattering matrix has the form

$$S(k) = 1 + c_k \mathcal{A}(k), \quad k \in \mathbb{R}_+ \setminus \sigma_d, \quad (3.11)$$

see e.g. [RS79, Ch XI.6], where σ_d is the set of zeros of the function D in $\overline{\mathbb{K}}_1$, and the "modified scattering amplitude" $\mathcal{A}(k)$ is given by

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_0 - \mathcal{A}_1, \\ \mathcal{A}_0 &= \psi_1 \psi_2, \quad \mathcal{A}_1 = \psi_1 Y \psi_2, \end{aligned} \quad (3.12)$$

where ψ_1, ψ_2 are defined by (3.1), (3.2).

Lemma 3.3. *Let $p, q \in \mathcal{H}_0$. Then the function $\mathcal{A}_0(k)$ is continuous in \mathbb{R}_+ , it has an analytic extension onto the whole complex plane and satisfies*

$$\mathcal{A}_0(k) = \frac{1}{\pi} \left(q_0 - 2p_0 k^2 - \int_0^\infty (2k^2 p(x) + q(x)) \cos 2kx dx \right) \quad \forall \quad k \in \mathbb{C}, \quad (3.13)$$

where $f_0 = \int_0^\infty f(x) dx$. The function $\mathcal{A}_1(k)$ is continuous in $\mathbb{R}_+ \setminus \sigma_d$ and it has a meromorphic extension onto the whole complex plane. Moreover, the functions $\mathcal{A}_0(k), \mathcal{A}_1(k)$ satisfy

$$\mathcal{A}_0(k) = e^{2\gamma \text{Im } k} O(k^2), \quad \mathcal{A}_0(ik) = e^{2\gamma \text{Re } k} O(k^2), \quad (3.14)$$

$$\mathcal{A}_1(k) = e^{2\gamma \operatorname{Im} k} O(k), \quad (3.15)$$

as $|k| \rightarrow \infty, k \in \overline{\mathbb{K}_1}$.

Proof. The operator-valued functions $\psi_1(k), \psi_2(k)$ are continuous in \mathbb{R}_+ and they have analytic extensions onto the whole complex plane. Then the function $\mathcal{A}_0(k)$ is continuous in \mathbb{R}_+ and it has an analytic extension onto the whole complex plane. The definitions (3.1), (3.2) and (3.12) give

$$\mathcal{A}_0(k) = \frac{2}{\pi} \int_0^\infty (-2k^2 p(x) \cos^2 kx + q(x) \sin^2 kx) dx,$$

which yields the identity (3.13). This identity implies the asymptotics (3.14).

Due to Lemma 2.1 ii) the function $\mathcal{A}_1(k)$ is continuous in $\mathbb{R}_+ \setminus \sigma_d$ and it has a meromorphic extension onto the whole complex plane. The definition (3.12) and the estimates (2.24) and (3.4) give the asymptotics (3.15). ■

3.3. Properties of the determinant. Lemma 2.1 i) shows that $Y_0(k) \in \mathcal{B}_1$, then the determinant $D(k) = \det(I + Y_0(k)), k \in \mathbb{K}_1$ is well defined.

Lemma 3.4. *Let $p, q \in \mathcal{H}_0$. Then*

- i) *The determinant $D(k) = \det(I + Y_0(k))$ is analytic in \mathbb{K}_1 and has an analytic extension from \mathbb{K}_1 onto the whole complex plane without zero, such that the function $kD(k)$ is entire.*
- ii) *The function $D(k)$ is real on the line $e^{i\frac{\pi}{4}}\mathbb{R}$.*

Proof. i) Due to Lemma 2.1 i) the operator-valued function $Y_0(k)$, and then the determinant $D(k)$, is analytic in $k \in \mathbb{K}_1$ and has an analytic extension from $k \in \mathbb{K}_1$ onto the whole complex plane without zero. It is proved in [BK16] that the function $kD(k)$ is entire.

ii) The identity (2.13) shows that $R_0(k)$ is real on the line $e^{i\frac{\pi}{4}}\mathbb{R}$, then $Y_0(k)$ is real also. Therefore, $D(k)$ is real on this line. ■

The estimates (2.23) give $\|Y_0(k)\|_{\mathcal{B}_1} = O(k^{-1})$ as $k \rightarrow e^{i\frac{\pi}{4}}\infty$. We can define the branch $\log D(k)$ for $k \in \mathbb{K}_1$ and $|k|$ large enough, by

$$\log D(k) = o(1) \quad \text{as} \quad k \rightarrow e^{i\frac{\pi}{4}}\infty.$$

We need the following standard results.

Lemma 3.5. *Let $p, q \in \mathcal{H}_0$. Then the following identity holds true:*

$$\operatorname{Tr} Y_0(k) = -\frac{(1+i)p_0 - i\widehat{p}(k) - \widehat{p}(ik)}{2k} - \frac{(1-i)q_0 + i\widehat{q}(k) - \widehat{q}(ik)}{4k^3}, \quad (3.16)$$

for any $k \in \mathbb{K}_1$, where $\widehat{f}(k) = \int_0^\infty e^{2ikx} f(x) dx$. Moreover, the function $\log D(k)$ satisfies

$$|\log D(k) + \sum_{n=1}^N \frac{1}{n} \operatorname{Tr}(-Y_0(k))^n| \leq \frac{C_1}{|k|^{N+1}}, \quad \forall \quad N \geq 1, \quad (3.17)$$

$$\log D(k) = -\sum_{n=1}^\infty \frac{1}{n} \operatorname{Tr}(-Y_0(k))^n, \quad (3.18)$$

for any $k \in \mathbb{K}_1, |k|$ large enough, and for some constant $C_1 > 0$, where the series converges absolutely and uniformly on any compact subset of \mathbb{K}_1 . Furthermore, the function $\log D$ satisfies

the asymptotics

$$\log D(k) = -\frac{(1+i)p_0 + o(1)}{2k} \quad \text{as } |k| \rightarrow \infty, \quad k \in \overline{\mathbb{K}}_1 \quad (3.19)$$

uniformly in $\arg k \in [0, \frac{\pi}{2}]$.

Proof. Let $k \in \mathbb{K}_1$. The identities (2.12), (2.19) imply

$$\text{Tr } Y_0(k) = \int_0^\infty \left(- (r_0(x, x, k) + r_0(x, x, ik))p(x) + \frac{(r_0(x, x, k) - r_0(x, x, ik))q(x)}{2k^2} \right) dx.$$

Then the identity (2.6) gives (3.16). The estimate (2.22) gives

$$|\text{Tr}(Y_0(k))^n| \leq \|Y_0(k)\|_{\mathcal{B}_2}^n \leq \left(\frac{C}{|k|} \right)^n, \quad n \geq 2 \quad (3.20)$$

for some constant $C > 0$. Then the series (3.18) converges absolutely and uniformly and it is well-known that the sum is equal to $\log D(k)$ (see [RS78, Lm XIII.17.6]). The estimates (3.20) imply (3.17). The relations (3.16), (3.17) give the asymptotics (3.19). ■

3.4. Identities for the determinant and S-matrix. We will determine asymptotics of the determinant in the complex plane. In the case of the Schrödinger operator it is sufficiently to obtain the asymptotics of the determinant $d(k)$ and the scattering matrix $s(k)$ in \mathbb{C}_+ . Then using the identity (2.11) we obtain the asymptotics in \mathbb{C}_- . In the case of fourth order operators the similar arguments give the asymptotics of the determinant in the domains $\mathbb{K}_1, \mathbb{K}_2$ (and, by the symmetry, in \mathbb{K}_4). In order to obtain the asymptotics in the domain \mathbb{K}_3 we need some additional analysis, more complicated than for $\mathbb{K}_1, \mathbb{K}_2$. The corresponding analysis for third order operators was carried out in [K16].

Introduce the 2×2 matrix-valued function $\Omega(k), k \in \mathbb{K}_1$ by

$$\Omega = 1 + c_k(\Omega_0 - \Omega_1), \quad \Omega_0 = \Psi_1 \Psi_2, \quad \Omega_1 = \Psi_1 Y \Psi_2, \quad (3.21)$$

where Ψ_1, Ψ_2 are defined by (3.3). The function Ω_0 has an analytic extension from \mathbb{K}_1 into the whole complex plane and the functions Ω_1 and Ω have meromorphic extensions from \mathbb{K}_1 into the whole complex plane.

The identity (3.22) below is similar to the relation (2.11) for Schrödinger operator. It gives an exact formula for an analytic extension of the determinant D in the domain \mathbb{K}_2 . In order to get the analytic extension in the domain \mathbb{K}_3 we use the identity (3.23). It is a crucial point for our consideration.

Lemma 3.6. *Let $p, q \in \mathcal{H}_0$, and let $k \in \mathbb{C} \setminus \{0\}$. Then the determinant D satisfies*

$$D(ik) = D(k)S(k), \quad (3.22)$$

$$D(-k) = D(k) \det \Omega(k), \quad (3.23)$$

where the function Ω is given by the definition (3.21), and the S-matrix $S(k)$ is continuous on \mathbb{R}_+ .

Proof. The identities (3.6) and (2.21) give

$$D(ik) = \det(1 + Y_0(ik)) = \det(1 + Y_0(k) + P_1(k)) = D(k) \det(1 + (1 - Y(k))P_1(k)).$$

The definition (3.5) yields

$$D(ik) = D(k) \det \left(1 + c_k (1 - Y(k)) \psi_2(k) \psi_1(k) \right).$$

Then the definitions (3.11), (3.12) and the identity (2.2) imply the identity (3.22).

Similarly, the identities (3.7) and (2.21) give

$$D(-k) = \det (1 + Y_0(-k)) = \det (1 + Y_0(k) + P_2(k)) = D(k) \det \left(1 + (1 - Y(k)) P_2(k) \right).$$

The definition (3.5) yields

$$D(-k) = D(k) \det \left(1 + c_k (1 - Y(k)) \Psi_2(k) \Psi_1(k) \right).$$

Then the identity (2.2) implies the identity (3.23).

Due to Lemma 3.3, the S -matrix $S(k)$ is continuous in $k \in \mathbb{R}_+ \setminus \sigma_d$ and it has a meromorphic extension from \mathbb{R}_+ onto \mathbb{C} . Moreover, if $k \in \sigma_d \cap \mathbb{R}_+$, then k is a zero of the functions $D(ik)$ and $D(k)$ of the same multiplicity. Due to the identity (3.22), $S(k)$ is continuous at the point $k \in \sigma_d$. Therefore, $S(k)$ is continuous on \mathbb{R}_+ . ■

Lemma 3.7. *Let $p, q \in \mathcal{H}_0$. Then the function $\Omega_0 = \Psi_1 \Psi_2$ satisfies the identity*

$$\Omega_0(k) = \begin{pmatrix} i\mathcal{A}_0(ik) & \mathcal{B}(k) \\ i\mathcal{B}(k) & \mathcal{A}_0(k) \end{pmatrix}, \quad k \in \mathbb{C}, \quad (3.24)$$

where \mathcal{A}_0 is defined by (3.12) and

$$\mathcal{B}(k) = \frac{2}{\pi} \int_0^\infty \left(-i2k^2 p(x) \operatorname{ch} kx \cos kx + iq(x) \operatorname{sh} kx \sin kx \right) dx. \quad (3.25)$$

The function \mathcal{B} satisfies

$$\mathcal{B}(k) = e^{\gamma(\operatorname{Re} k + \operatorname{Im} k)} O(k^2) \quad (3.26)$$

as $|k| \rightarrow \infty, k \in \overline{\mathbb{K}}_1$.

Proof. We prove the identity (3.24). The definition (3.3) implies

$$\Omega_0(k) = \Psi_1(k) \Psi_2(k) = \begin{pmatrix} i\psi_1(ik)\psi_2(ik) & \psi_1(ik)\psi_2(k) \\ i\psi_1(k)\psi_2(ik) & \psi_1(k)\psi_2(k) \end{pmatrix}. \quad (3.27)$$

The definitions (3.1), (3.2) and (3.25) give

$$\psi_1(ik)\psi_2(k) = \frac{2}{\pi} \int_0^\infty \left(-i2k^2 p(x) \operatorname{ch} kx \cos kx + iq(x) \operatorname{sh} kx \sin kx \right) dx = \mathcal{B}(k).$$

and similarly $\psi_1(k)\psi_2(ik) = \mathcal{B}(k)$. Substituting these identities and the definition (3.12) into (3.27) we obtain the identity (3.24). The definition (3.25) yields the asymptotics (3.26). ■

4. PROOF OF THE MAIN THEOREMS AND TRACE FORMULAS IN TERMS OF RESONANCES

4.1. Asymptotics of the determinant. We prove our preliminary Proposition 1.1 which gives some properties of the determinant and its asymptotics in the domain \mathbb{K}_1 .

Proof of Proposition 1.1. i) The statement is proved in Lemma 2.1 i).

ii) Due to Lemma 3.4, the function D has an analytic extension from \mathbb{K}_1 onto $\mathbb{C} \setminus \{0\}$, it is real on the line $e^{i\frac{\pi}{4}}\mathbb{R}$ and the function $kD(k)$ is entire. The asymptotics (3.19) yields (1.9). The function $D(k)$ has a finite number of zeros in $\overline{\mathbb{K}}_1$, then the operator H has a finite number of eigenvalues. ■

The asymptotics of $D(k)$ in the domain \mathbb{K}_1 is known due to (1.9). We analyze the function $D(k)$ in the domains $\mathbb{K}_2, \mathbb{K}_3$ by the following way. We obtain the asymptotics of $S(k)$ and $\Omega(k)$ in \mathbb{K}_1 . Then we use the identities (3.22), (3.23) in order to determine the asymptotics of $D(ik), D(-k)$ in \mathbb{K}_1 , which gives the asymptotics of $D(k)$ in $\mathbb{K}_2, \mathbb{K}_3$.

Lemma 4.1. *Let $p, q \in \mathcal{H}_0$ and let $k \in \overline{\mathbb{K}}_1, |k| \rightarrow \infty$. Then*

$$S(k) = 1 + e^{2\gamma \operatorname{Im} k} O(k^{-1}), \quad (4.1)$$

$$D(ik) = 1 + e^{2\gamma \operatorname{Im} k} O(k^{-1}), \quad (4.2)$$

$$D(-k) = e^{2\gamma \operatorname{Im} k} O(k^{-1}) + e^{2\gamma \operatorname{Re} k} O(k^{-1}) + e^{2\gamma(\operatorname{Re} k + \operatorname{Im} k)} O(k^{-2}) \quad (4.3)$$

uniformly in $\arg k \in [0, \frac{\pi}{2}]$.

Proof. Let $|k| \rightarrow \infty, k \in \overline{\mathbb{K}}_1$. Substituting the asymptotics (3.14) and (3.15) into the definition (3.12) we obtain $\mathcal{A}(k) = e^{2\gamma \operatorname{Im} k} O(k^2)$. Then the identity (3.11) gives the asymptotics (4.1). Substituting the asymptotics (1.9), (4.1) into (3.22) we obtain the asymptotics (4.2).

Substituting the asymptotics (3.14) and (3.26) into the identity (3.24) we obtain

$$\Omega_0(k) = \begin{pmatrix} e^{2\gamma \operatorname{Re} k} O(k^2) & e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(k^2) \\ e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(k^2) & e^{2\gamma \operatorname{Im} k} O(k^2) \end{pmatrix}. \quad (4.4)$$

The definitions (3.3), (3.21) give

$$\Omega_1(k) = \begin{pmatrix} i\psi_1(ik)Y(k)\psi_2(ik) & \psi_1(ik)Y(k)\psi_2(k) \\ i\psi_1(k)Y(k)\psi_2(ik) & \psi_1(k)Y(k)\psi_2(k) \end{pmatrix}$$

Then the estimates (2.24) and (3.4) imply

$$\Omega_1(k) = \begin{pmatrix} e^{2\gamma \operatorname{Re} k} O(k) & e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(k) \\ e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(k) & e^{2\gamma \operatorname{Im} k} O(k) \end{pmatrix}. \quad (4.5)$$

Substituting the asymptotics (4.4) and (4.5) into the definition (3.21) we obtain

$$\Omega(k) = \begin{pmatrix} 1 + e^{2\gamma \operatorname{Re} k} O(k^{-1}) & e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(k^{-1}) \\ e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(k^{-1}) & 1 + e^{2\gamma \operatorname{Im} k} O(k^{-1}) \end{pmatrix},$$

which yields

$$\det \Omega(k) = e^{2\gamma \operatorname{Im} k} O(k^{-1}) + e^{2\gamma \operatorname{Re} k} O(k^{-1}) + e^{2\gamma(\operatorname{Re} k + \operatorname{Im} k)} O(k^{-2}). \quad (4.6)$$

Substituting the asymptotics (1.9), (4.6) into the identity (3.23) we obtain the asymptotics (4.3). ■

4.2. Asymptotics of the counting function. Recall that the function $D(k)$ is analytic in $\mathbb{C} \setminus \{0\}$ and may have a simple pole at the point $k = 0$. We prove our main results.

Proof of Theorem 1.2. The asymptotics (1.9), (4.2), (4.3) give the estimate (1.10) in $\mathbb{K}_1, \mathbb{K}_2, \mathbb{K}_3$. The symmetry $D(k) = \overline{D(\bar{k})}$ imply the estimate (1.10) in \mathbb{K}_4 .

The asymptotics (4.2) gives $D(k) = 1 + e^{-2\gamma \operatorname{Re} k} O(k^{-1})$ as $|k| \rightarrow \infty, k \in \mathbb{K}_2$. This yields $|k(D(k) - 1)| \leq C e^{-2\gamma \operatorname{Re} k}$ for all $k \in \mathbb{K}_2$ with modulus large enough. Let $k_* \in \mathbb{K}_2$ be a resonance. Then the identity $D(k_*) = 0$ gives the estimate (1.12).

We prove the estimate (1.11). Recall that the function $D(k)$ is analytic in $\mathbb{C} \setminus \{0\}$ and may have a simple pole at the point $k = 0$. Let the function $F(k) = k^m D(k), m \leq 1$, be entire and satisfy $F(0) \neq 0$. Let $\mathcal{N}_F(r)$ be the number of zeros of the function F in the disc $|k| < r$ counted with multiplicity. If $D(0) \neq 0$, then $\mathcal{N} = \mathcal{N}_F$, if $k = 0$ is a zero of D of multiplicity ℓ , then $\mathcal{N} = \mathcal{N}_F + \ell$. It is sufficiently to prove the estimate (1.11) for \mathcal{N}_F . The estimate (1.10) gives

$$\log |F(k)| \leq \gamma((\operatorname{Re} k)_- + (\operatorname{Im} k)_-) + C \log |k| \quad (4.7)$$

for all $k \in \mathbb{C}, |k|$ large enough and for some $C > 0$. Substituting the estimate (4.7) into Jensen's formula

$$\int_0^r \frac{\mathcal{N}_F(t)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \log |F(re^{i\theta})| d\theta - \log |F(0)|, \quad (4.8)$$

we obtain

$$\int_0^r \frac{\mathcal{N}_F(t)}{t} dt \leq -\frac{\gamma r}{\pi} \left(\int_{\frac{\pi}{2}}^{\pi} \cos \theta d\theta + \int_{\pi}^{\frac{3\pi}{2}} (\cos \theta + \sin \theta) d\theta + \int_{\frac{3\pi}{2}}^{2\pi} \sin \theta d\theta \right) + C \log r = \frac{4\gamma r}{\pi} + C \log r$$

for all $r > 0$ large enough. Then there exists

$$\lim_{r \rightarrow +\infty} \frac{1}{r} \int_0^r \frac{\mathcal{N}_F(t)}{t} dt \leq \frac{4\gamma}{\pi}.$$

The estimate (1.11) follows from the following well known result, see, e.g., [L71, Lm II.4.3]:

Let $\mathcal{N}_F(t)$ be non-decreasing function on \mathbb{R}_+ , $\mathcal{N}_F(t) = 0$ as $0 \leq t < \varepsilon$ for some $\varepsilon > 0$, and let the function

$$I(r) = \frac{1}{r} \int_0^r \frac{\mathcal{N}_F(t)}{t} dt, \quad r \in \mathbb{R}_+,$$

has the limit as $r \rightarrow \infty$. Then $\mathcal{N}_F(r) = r(I(r) + o(1))$. ■

4.3. Scattering phase and trace formulas. Now we discuss the Hadamard factorization for the Fredholm determinant D . The function $kD(k)$ is entire, then

$$D(k) = \frac{\alpha}{k^m} (1 + \beta k + O(k^2)) \quad \text{as } |k| \rightarrow 0, \quad m \leq 1,$$

for some $\alpha, \beta \in \mathbb{C}$. Let $\zeta_n, n \in \mathbb{N}$, be the zeros of the function D in $\mathbb{C} \setminus \{0\}$ labeled by $0 < |\zeta_1| \leq |\zeta_2| \leq \dots$ counting with multiplicities. The estimate (1.10) provides the standard Hadamard factorization

$$D(k) = \frac{\alpha}{k^m} e^{\beta k} \lim_{r \rightarrow \infty} \prod_{|\zeta_n| < r} \left(1 - \frac{k}{\zeta_n}\right) e^{\frac{k}{\zeta_n}}, \quad (4.9)$$

absolutely and uniformly on any compact subset in $\mathbb{C} \setminus \{0\}$.

Remark. It is proved in [BK16] that $\beta = (i - 1)\gamma$ in the case $(p, q) \in \mathcal{H}_1 \times \mathcal{H}_0, p(\gamma - 0) \neq 0$.

The S-matrix $S(k)$, $k \in \mathbb{R}_+$ is a complex number and $|S(k)| = 1$. Thus we have

$$S(k) = e^{-2\pi i \phi_{sc}(k)}, \quad k \in \mathbb{R}_+, \quad (4.10)$$

where $\phi_{sc}(k)$ is a scattering phase. The function $S(k)$ is continuous on \mathbb{R}_+ and the asymptotics (4.1) shows that $S(k) = 1 + O(k^{-1})$ as $k \rightarrow +\infty$. If we assume that the function $\phi_{sc}(k)$ is also continuous on \mathbb{R}_+ , then formula (4.10) uniquely determines $\phi_{sc}(k)$, $k > 0$, by $\phi_{sc}(k) = \frac{i}{2\pi} \log S(k)$ and the asymptotics $\phi_{sc}(k) = O(k^{-1})$ as $k \rightarrow +\infty$.

Our next results concern the trace formula in terms of resonances. Trace formulas for one-dimensional Schrödinger operators in terms of resonances were determined in [K04] and for third order operators in [K16]. Here we use the approach from [K04].

Theorem 4.2. *Let $(p, q) \in \mathcal{H}_0$. Then the following trace formulas hold true:*

$$4k^4 \operatorname{Tr}(R_0(k) - R(k)) = -m + \beta k + k^2 \lim_{r \rightarrow \infty} \sum_{|\zeta_n| < r} \frac{1}{\zeta_n(k - \zeta_n)}, \quad k \in \mathbb{K}_1 \setminus \sigma_d, \quad (4.11)$$

the series converges absolutely and uniformly on any compact subset in $\mathbb{K}_1 \setminus \sigma_d$,

$$\phi'_{sc}(k) = \frac{1}{2\pi i} \left((1-i)\beta + \sum_{n=1}^{\infty} \frac{k}{\zeta_n} \left(\frac{1}{ik - \zeta_n} + \frac{1}{k - \zeta_n} \right) \right), \quad k \in \mathbb{R}_+ \setminus \sigma_d, \quad (4.12)$$

the series converges absolutely and uniformly on any compact subset in $\mathbb{R}_+ \setminus \sigma_d$.

Proof. Let $k \in \mathbb{K}_1 \setminus \sigma_d$. The definitions (1.5), (2.13) show that the operators $VR_0(k)$, $V_2R_0(k)$ and $R_0(k)V_1$ are Hilbert-Schmidt. Then the operator

$$R_0(k) - R(k) = R_0(k)VR(k) = R_0(k)VR_0(k) - R_0(k)VR_0(k)VR(k)$$

is trace class. Due to the identities (2.2), (2.5), (2.21) and $Y'_0(k) = 4k^3V_2R_0^2(k)V_1$, the derivative of D satisfies

$$\frac{1}{4k^3} \frac{D'(k)}{D(k)} = \operatorname{Tr}((I + Y_0(k))^{-1}V_2R_0^2(k)V_1) = \operatorname{Tr} R_0(k)VR(k) = \operatorname{Tr}(R_0(k) - R(k)). \quad (4.13)$$

The identity (4.9) gives

$$\frac{D'(k)}{D(k)} = \beta - \frac{m}{k} + k \lim_{r \rightarrow \infty} \sum_{|\zeta_n| < r} \frac{1}{\zeta_n(k - \zeta_n)}. \quad (4.14)$$

This identity together with (4.13) yields the trace formula (4.11).

The function $S(k)$ is continuous in \mathbb{R}_+ , has a meromorphic extension onto the whole complex plane and, due to equations (4.10) and (3.22), it satisfies the identities

$$e^{-2\pi i \phi_{sc}(k)} = S(k) = \frac{D(ik)}{D(k)}, \quad \forall \quad k > 0.$$

Differentiating this identity we obtain

$$-2\pi i \phi'_{sc}(k) = i \frac{D'(ik)}{D(ik)} - \frac{D'(k)}{D(k)}.$$

Then equation (4.14) implies (4.12). ■

5. EULER-BERNOULLI OPERATORS AND PROOF OF THEOREM 1.5

5.1. The Liouville type transformation. We consider the Euler-Bernoulli operator

$$\mathcal{E}u = \frac{1}{b}(au'')'', \quad (5.1)$$

acting on $L^2(\mathbb{R}_+, b(x)dx)$, with the boundary conditions

$$u = 0, \quad \text{and} \quad u'' + \frac{a'}{5a}u' = 0 \quad \text{at} \quad x = 0, \quad (5.2)$$

where the coefficients a, b are positive, $a = b = 1$ outside the unit interval and satisfy

$$a - 1, b - 1 \in \mathcal{H}_4, \quad \left(\frac{3a'}{a} + \frac{5b'}{b} \right) \Big|_{x=0} = 0. \quad (5.3)$$

Now we consider the Liouville type transformation of the operator \mathcal{E} into the operator H , defined by (1.1), (1.2) with specific p, q depending on a, b . In order to define this transformation we introduce the new variable $t \in \mathbb{R}_+$ by

$$t = t(x) = \int_0^x \left(\frac{b(s)}{a(s)} \right)^{\frac{1}{4}} ds, \quad \forall \quad x \in \mathbb{R}_+. \quad (5.4)$$

Let $x = x(t)$ be the inverse function for $t(x), x \in \mathbb{R}_+$. Introduce the unitary transformation $U : L^2(\mathbb{R}_+, b(x)dx) \rightarrow L^2(\mathbb{R}_+, dt)$ by

$$u(x) \mapsto (Uu)(t) = a^{\frac{1}{8}}(x(t))b^{\frac{3}{8}}(x(t))u(x(t)). \quad (5.5)$$

Introduce the functions $\alpha(t), \beta(t), t \in \mathbb{R}_+$, by

$$\alpha(t) = \frac{1}{a(x(t))} \frac{da(x(t))}{dt}, \quad \beta(t) = \frac{1}{b(x(t))} \frac{db(x(t))}{dt}. \quad (5.6)$$

Then the functions $\alpha, \beta \in L^1(\mathbb{R}_+)$ are real, compactly supported and satisfy

$$\ddot{\alpha}, \ddot{\beta} \in L^1(\mathbb{R}_+), \quad \text{where} \quad \dot{f} = \frac{df}{dt}.$$

Let the operator H be defined by (1.1), where the coefficients $p(t), q(t), t \in \mathbb{R}_+$, have the forms

$$p = -\frac{\dot{\eta}_0 + \varkappa}{2}, \quad (5.7)$$

$$\varkappa = \frac{5\alpha^2 + 5\beta^2 + 6\alpha\beta}{32}, \quad \eta_0 = \frac{3\alpha + 5\beta}{4}, \quad (5.8)$$

and

$$q = \frac{d}{dt}((\dot{\eta}_2 + \eta_2^2)\eta_1 - \ddot{\eta}_1) + ((\dot{\eta}_2 + \eta_2^2)\eta_1 - \ddot{\eta}_1)\eta_1, \quad (5.9)$$

$$\eta_1 = \frac{\alpha + 3\beta}{8}, \quad \eta_2 = \frac{3\alpha + \beta}{8}.$$

Note that the coefficients $(p, q) \in \mathcal{H}_2 \times \mathcal{H}_0$, where γ is given by the definition (1.19). The definition (5.8) shows that

$$\varkappa \geq \frac{\alpha^2 + \beta^2}{16} \geq 0, \quad (5.10)$$

moreover, $\varkappa = 0$ iff $\alpha = \beta = 0$. The proof of Theorem 1.5 is based on this observation.

Let the coefficients a, b be positive and satisfy the conditions (5.3). Let the operator \mathcal{E} be defined by (5.1) and let the operator H be defined by (1.1), where the coefficients p, q have

the forms (5.7), (5.9). Repeating the arguments from [BK15] we obtain that the operators \mathcal{E} and H are unitarily equivalent and satisfy:

$$\mathcal{E} = U^{-1} H U. \quad (5.11)$$

where the operator U is defined by (5.5).

Proof of Corollary 1.4. The definitions (5.4), (5.7), (5.9) show that $(p, q) \in \mathcal{H}_2 \times \mathcal{H}_0$, where γ is given by (1.19). This yields the statement. ■

The following Lemma is a corollary of Proposition 1.1. Here we determine asymptotics of the determinant in the domain \mathbb{K}_1 , which is crucial for the proof of Theorem 1.5.

Corollary 5.1. *Let the coefficients a, b be positive and satisfy the conditions (5.3). Then the determinant $D(k)$ satisfies*

$$D(k) = 1 + \frac{1+i}{8k} \int_0^\infty \varkappa(t) dt + \frac{O(1)}{k^2} \quad \text{as} \quad |k| \rightarrow \infty, \quad k \in \overline{\mathbb{K}}_1 \quad (5.12)$$

uniformly in $\arg k \in [0, \frac{\pi}{2}]$, where $\varkappa(t)$ is given by the definition (5.8).

Proof. Due to the last condition in (1.18) we have $\eta_0(0) = 0$. Identity (5.7) gives

$$p_0 = \int_0^\infty p(t) dt = -\frac{1}{2} \int_0^\infty \varkappa(t) dt.$$

Substituting this identities into the asymptotics (1.9) we obtain the asymptotics (5.12). ■

Proof of Theorem 1.5. The proof uses the arguments of Isozaki and Korotyaev [IK12]. Assume that the operator \mathcal{E} does not have any eigenvalues and resonances. The identity (4.9) shows that $D(k) = k^{-m} \alpha e^{\beta k}$ in this case, where $\alpha, \beta \in \mathbb{C}$ and $m = 0$ or 1 . The asymptotics (1.9) implies $D = 1$. Then the second term in the asymptotics (5.12) vanishes, which yields $\int_0^\infty \varkappa(t) dt = 0$. Due to the estimate (5.10), we have $\varkappa = 0$ and $\alpha = \beta = 0$ in this case, then $a = b = 1$ on \mathbb{R}_+ .

Conversely, let $a = b = 1$ on \mathbb{R}_+ . Then the operator \mathcal{E} , given by (5.1), has the form $\mathcal{E} = \partial^4$. The definitions (1.6), (1.7) show that $D = 1$ in this case. Then, due to the identity (4.9), there are not any eigenvalues and resonances. ■

6. RESONANCES FOR COEFFICIENTS WITH JUMP DISCONTINUITY AND PROOF OF THEOREM 1.3

6.1. Asymptotics of auxiliary functions. The function $D(k)$ has a finite number of zeros in the domain \mathbb{K}_1 . The identity (3.22) shows that ik with large $|k|$ is a resonance in \mathbb{K}_2 iff k is a zero of the function $S(k)$ in \mathbb{K}_1 . Thus in order to determine asymptotics of resonances in \mathbb{K}_2 we need to improve asymptotics of the scattering matrix $S(k)$ in \mathbb{K}_1 . Similarly, the identity (3.23) shows that $-k$ with large $|k|$ is a resonance in \mathbb{K}_3 iff k is a zero of the function $\det \Omega(k)$ in \mathbb{K}_1 . Then in order to determine asymptotics of resonances in \mathbb{K}_3 we have to improve asymptotics of the function $\det \Omega(k)$ in \mathbb{K}_1 . Moreover, due to the symmetry of the determinant it is sufficiently to consider in this case the domain

$$\mathbb{K}_1^+ = \{k \in \mathbb{C} : \arg k \in (0, \frac{\pi}{4})\}.$$

In the following Lemma we improve the asymptotics of the functions \mathcal{A}_0 and \mathcal{B} , given by (3.12) and (3.25) respectively.

Lemma 6.1. *Let $(p, q) \in \mathcal{H}_1 \times \mathcal{H}_0$. Then the functions \mathcal{A}_0 and \mathcal{B} satisfy*

$$\mathcal{A}_0(k) = -\frac{ik e^{-i2k\gamma}}{2\pi} (p_+ + o(1)) + O(k^2), \quad (6.1)$$

$$\mathcal{A}_0(ik) = \frac{k e^{2k\gamma}}{2\pi} (p_+ + o(1)) + O(k^2), \quad (6.2)$$

as $|k| \rightarrow \infty, k \in \overline{\mathbb{K}}_1$ uniformly in $\arg k \in [0, \frac{\pi}{2}]$,

$$\mathcal{B}(k) = -\frac{ik}{2\pi} (1+i) e^{(1-i)k\gamma} (p_+ + e^{-2\gamma \operatorname{Im} k} O(1) + o(1)) + O(k^2), \quad (6.3)$$

as $|k| \rightarrow \infty, k \in \overline{\mathbb{K}}_1^+$ uniformly in $\arg k \in [0, \frac{\pi}{4}]$.

Proof. Let $|k| \rightarrow \infty, k \in \overline{\mathbb{K}}_1$. The identity (3.13) implies

$$\mathcal{A}_0(k) = -\frac{k^2}{\pi} \int_0^\infty p(x) e^{-i2kx} dx + O(k^2) + e^{2\gamma \operatorname{Im} k} O(1).$$

Integrating by parts we obtain (6.1). Similarly,

$$\mathcal{A}_0(ik) = \frac{k^2}{\pi} \int_0^\infty p(x) e^{2kx} dx + O(k^2) + e^{2\gamma \operatorname{Re} k} O(1),$$

which implies (6.2).

Let $k \in \overline{\mathbb{K}}_1^+, |k| \rightarrow \infty$. The definition (3.25) gives

$$\mathcal{B}(k) = -\frac{ik^2}{\pi} \int_0^\infty p(x) (e^{(1-i)kx} + e^{(1+i)kx}) dx + O(k^2) + e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(1).$$

Integrating by parts we obtain the asymptotics (6.3). ■

Now we improve the asymptotics of the functions \mathcal{A}_1 and Ω_1 , given by (3.12) and (3.21) respectively.

Lemma 6.2. *Let $(p, q) \in \mathcal{H}_1 \times \mathcal{H}_0$. Then the functions \mathcal{A}_1 and Ω_1 satisfy*

$$\mathcal{A}_1(k) = e^{2\gamma \operatorname{Im} k} O(1), \quad (6.4)$$

$$\Omega_1(k) = \begin{pmatrix} e^{2\gamma \operatorname{Re} k} O(1) & e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(1) \\ e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(1) & e^{2\gamma \operatorname{Im} k} O(1) \end{pmatrix} \quad (6.5)$$

as $|k| \rightarrow \infty, k \in \overline{\mathbb{K}}_1$ uniformly in $\arg k \in [0, \frac{\pi}{2}]$.

Proof. The definitions (3.3), (3.12), (3.21) and the estimates (2.25), (3.4) give

$$\begin{aligned} \mathcal{A}_1(k) &= \psi_1(k) Y_0(k) \psi_2(k) + e^{2\gamma \operatorname{Im} k} O(1), \\ \Omega_1(k) &= \Psi_1(k) Y_0(k) \Psi_2(k) + \begin{pmatrix} e^{2\gamma \operatorname{Re} k} O(1) & e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(1) \\ e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(1) & e^{2\gamma \operatorname{Im} k} O(1) \end{pmatrix}, \end{aligned} \quad (6.6)$$

as $|k| \rightarrow \infty, k \in \overline{\mathbb{K}}_1$.

Let $k \in \overline{\mathbb{K}}_1$. The definitions (3.1), (3.2) and the identity (2.19) imply

$$\psi_1(k) Y_0(k) \psi_2(k) = k^2 a_1(k) - k(a_2(k) + a_3(k)) + a_4(k),$$

where

$$a_1 = 4 \int_0^\infty p(x) \cos kx \left(\int_0^\infty \frac{\partial R_0(x, y, k)}{\partial x} (p(y) \cos ky)' dy \right) dx,$$

$$\begin{aligned}
 a_2 &= 2 \int_0^\infty q(x) \sin kx \left(\int_0^\infty R_0(x, y, k) (p(y) \cos ky)' dy \right) dx, \\
 a_3 &= 2 \int_0^\infty p(x) \cos kx \left(\int_0^\infty \frac{\partial R_0(x, y, k)}{\partial x} q(y) \sin ky dy \right) dx, \\
 a_4 &= \int_0^\infty q(x) \sin kx \left(\int_0^\infty R_0(x, y, k) q(y) \sin ky dy \right) dx.
 \end{aligned}$$

Using the identity $a_3 = -a_2$ we obtain

$$\psi_1(k)Y_0(k)\psi_2(k) = k^2 a_1(k) + a_4(k). \quad (6.7)$$

Let $|k| \rightarrow \infty, k \in \overline{\mathbb{K}}_1$. The identity (2.13) gives

$$a_4(k) = \frac{e^{2\gamma \operatorname{Im} k} O(1)}{k^3}. \quad (6.8)$$

Moreover, the identity (2.12) yields

$$a_1(k) = b_1(k) + b_2(k), \quad (6.9)$$

where

$$\begin{aligned}
 b_1(k) &= -2 \int_0^\infty p(x) \cos kx \int_0^\infty r_0(x, y, k) p(y) \cos ky dy dx, \\
 b_2(k) &= -2 \int_0^\infty p(x) \cos kx \int_0^\infty r_0(x, y, ik) p(y) \cos ky dy dx.
 \end{aligned}$$

The identity (2.6) and the integration by parts give

$$b_1(k) = -\frac{i}{k} \int_0^\infty p(x) \cos kx \left(\int_0^\infty (e^{ik|x-y|} - e^{ik(x+y)}) p(y) \cos ky dy \right) dx = \frac{e^{2\gamma \operatorname{Im} k} O(1)}{k^2},$$

and similarly,

$$b_2(k) = \frac{e^{2\gamma \operatorname{Im} k} O(1)}{k^2}.$$

Then the identity (6.9) yields

$$a_1(k) = \frac{e^{2\gamma \operatorname{Im} k} O(1)}{k^2}. \quad (6.10)$$

Substituting the asymptotics (6.8), (6.10) into the identity (6.7) we obtain

$$\psi_1(k)Y_0(k)\psi_2(k) = e^{2\gamma \operatorname{Im} k} O(1).$$

The asymptotics (6.6) gives the asymptotics (6.4).

The similar arguments show that

$$\begin{aligned}
 \psi_1(ik)Y_0(k)\psi_2(ik) &= e^{2\gamma \operatorname{Re} k} O(1), \\
 \psi_1(ik)Y_0(k)\psi_2(k) &= e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(1), \\
 \psi_1(k)Y_0(k)\psi_2(ik) &= e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(1).
 \end{aligned}$$

Substituting these asymptotics into the definition (3.3) we obtain

$$\Psi_1(k)Y_0(k)\Psi_2(k) = \begin{pmatrix} e^{2\gamma \operatorname{Re} k} O(1) & e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(1) \\ e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(1) & e^{2\gamma \operatorname{Im} k} O(1) \end{pmatrix}.$$

The asymptotics (6.6) gives the asymptotics (6.5). ■

6.2. Asymptotics of resonances. Now we determine the sharp asymptotics of the scattering matrix $S(k)$ and the function $\det \Omega(k)$ in \mathbb{K}_1 .

Lemma 6.3. *Let $(p, q) \in \mathcal{H}_1 \times \mathcal{H}_0$ and let $p_+ = p(\gamma - 0) \neq 0$. Then*

$$S(k) = 1 + O(k^{-1}) - \frac{p_+}{4k^2} e^{-i2k\gamma} (1 + o(1)), \quad (6.11)$$

as $k \in \overline{\mathbb{K}}_1, |k| \rightarrow \infty$, uniformly in $\arg k \in [0, \frac{\pi}{2}]$,

$$\det \Omega(k) = \frac{p_+}{4k^2} e^{2k\gamma} (1 + o(1)) + \left(\frac{p_+}{4k^2} \right)^2 e^{2(1-i)k\gamma} (1 + o(1)) \quad (6.12)$$

as $k \in \overline{\mathbb{K}}_1^+, |k| \rightarrow \infty$, uniformly in $\arg k \in [0, \frac{\pi}{4}]$.

Proof. Let $k \in \overline{\mathbb{K}}_1, |k| \rightarrow \infty$. Substituting the asymptotics (6.4) into the definitions (3.11)–(3.12) we obtain

$$S(k) = 1 + c_k \mathcal{A}_0 + \frac{e^{2\gamma \operatorname{Im} k} O(1)}{k^3}, \quad c_k = \frac{\pi}{i2k^3}. \quad (6.13)$$

Substituting the asymptotics (6.1) into (6.13) we obtain the asymptotics (6.11).

The definition (3.21) gives

$$\det \Omega = 1 + c_k \operatorname{Tr}(\Omega_0 - \Omega_1) + c_k^2 \det(\Omega_0 - \Omega_1). \quad (6.14)$$

Let $k \in \overline{\mathbb{K}}_1^+, |k| \rightarrow \infty$. The identity (3.24) and the asymptotics (6.5) give

$$\operatorname{Tr}(\Omega_0(k) - \Omega_1(k)) = i\mathcal{A}_0(ik) + \mathcal{A}_0(k) + e^{2\gamma \operatorname{Re} k} O(1).$$

The asymptotics (6.1) and (6.2) imply

$$\operatorname{Tr}(\Omega_0(k) - \Omega_1(k)) = \frac{ip_+k}{2\pi} e^{2k\gamma} \left(1 + o(1) + e^{2\gamma(\operatorname{Im} k - \operatorname{Re} k)} O(1) \right), \quad (6.15)$$

Moreover, the identity (3.24) and the asymptotics (6.5) yield

$$\begin{aligned} \det(\Omega_0(k) - \Omega_1(k)) &= (i\mathcal{A}_0(ik) + e^{2\gamma \operatorname{Re} k} O(1)) (\mathcal{A}_0(k) + e^{2\gamma \operatorname{Im} k} O(1)) \\ &\quad - i(\mathcal{B}(k) + e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(1))^2. \end{aligned} \quad (6.16)$$

The asymptotics (6.1), (6.2) and (6.3) give

$$\begin{aligned} i\mathcal{A}_0(ik) + e^{2\gamma \operatorname{Re} k} O(1) &= \frac{ip_+k}{2\pi} e^{2k\gamma} (1 + o(1)), \\ \mathcal{A}_0(k) + e^{2\gamma \operatorname{Im} k} O(1) &= -\frac{ip_+k}{2\pi} e^{-i2k\gamma} (1 + o(1) + e^{-2\gamma \operatorname{Im} k} O(k)), \\ \mathcal{B}(k) + e^{\gamma(\operatorname{Im} k + \operatorname{Re} k)} O(1) &= -\frac{ip_+k}{2\pi} (1 + i) e^{(1-i)k\gamma} (1 + o(1) + e^{-2\gamma \operatorname{Im} k} O(1)). \end{aligned}$$

Substituting these asymptotics into the relation (6.16) we obtain

$$\det(\Omega_0(k) - \Omega_1(k)) = -\left(\frac{p_+k}{2\pi} \right)^2 e^{2(1-i)k\gamma} (1 + o(1) + e^{-2\gamma \operatorname{Im} k} O(k)). \quad (6.17)$$

Substituting the asymptotics (6.15) and (6.17) into the identity (6.14) we obtain

$$\begin{aligned} \det \Omega(k) &= 1 + \frac{p_+ e^{2k\gamma}}{4k^2} (1 + o(1) + e^{2\gamma(\operatorname{Im} k - \operatorname{Re} k)} O(1)) \\ &\quad + \left(\frac{p_+}{4k^2} \right)^2 e^{2(1-i)k\gamma} (1 + o(1) + e^{-2\gamma \operatorname{Im} k} O(k)) \\ &= \left(\frac{p_+}{4k^2} \right)^2 e^{2(1-i)k\gamma} \left(1 + o(1) + \frac{4k^2}{p_+} e^{i2k\gamma} (1 + o(1)) \right), \end{aligned}$$

which yields the asymptotics (6.12). ■

We are ready to determine asymptotics of resonances.

Proof of Theorem 1.3. The function $D(k)$ has a finite number of zeros in the domain \mathbb{K}_1 .

Let $k \in \mathbb{K}_1$, $|k| \rightarrow \infty$ and let ik be a resonance. The identity (3.22) shows that k is a zero of the function $S(k)$ in \mathbb{K}_1 . The asymptotics (6.11) and the identity $S(k) = 0$ imply that k satisfies the equation

$$k^2 e^{i2k\gamma} = \frac{p_+}{4} (1 + o(1)).$$

Then k lies on the logarithmic curve Γ in \mathbb{K}_1 , given by

$$|k| = \frac{|p_+|^{\frac{1}{2}} e^{\gamma \operatorname{Im} k}}{2} (1 + o(1)),$$

and satisfies

$$ik = \frac{ij_n \pi}{\gamma} - \frac{\log k}{\gamma} + \frac{1}{2\gamma} \log \frac{|p_+|}{4} + o(1) \quad (6.18)$$

and there are no any other large resonances in $i\mathbb{K}_+$.

Let $k \in \mathbb{K}_1^+$, let $-k$ be a resonance and let $|k|$ be large enough. The identity (3.23) shows that $-k$ is a zero of the function $\det \Omega(k)$ in \mathbb{K}_1 . The identity $\det \Omega(k) = 0$ and the asymptotics (6.12) show that

$$k^2 e^{2ik\gamma} = -\frac{p_+}{4} (1 + o(1)).$$

Then k lies on the curve Γ and satisfies

$$-k = -\frac{(j_n + \frac{1}{2})\pi}{\gamma} - \frac{i \log k}{\gamma} + \frac{i}{2\gamma} \log \frac{|p_+|}{4} + o(1) \quad (6.19)$$

and there are no any other large resonances in $-\mathbb{K}_1^+$. The asymptotics (6.18), (6.19) give (1.15), which yields the asymptotics (1.16). ■

6.3. Further discussions. An entire function $f(z)$ is said to be of exponential type if there is a constant A such that $|f(z)| \leq \text{const } e^{A|z|}$ everywhere. The infimum of the set of A for which such inequality holds is called the type of f . For each exponential type function f we define the types $\rho_{\pm}(f)$ in \mathbb{C}_{\pm} by

$$\rho_{\pm}(f) \equiv \limsup_{y \rightarrow \infty} \frac{\log |f(\pm iy)|}{y}.$$

We introduce the class of exponential type functions. The function f is said to belong to the Cartwright class \mathcal{C}_{ρ} if f is entire, of exponential type, and the following conditions hold true:

$$\int_{\mathbb{R}} \frac{\log(1 + |f(x)|)}{1 + x^2} dx < \infty, \quad \rho_+(f) = 0, \quad \rho_-(f) = 2\rho > 0,$$

for some $\rho > 0$. We recall the Levinson Theorem (see [Ko88]): *Let the entire function $f \in \mathcal{C}_\rho$. Let $\mathcal{N}(r)$ be the number of zeros of the function f in the disc $|k| < r$, counted with multiplicity. Then $\mathcal{N}(r, f) = \frac{2\rho}{\pi}r + o(r)$ as $r \rightarrow \infty$.*

We will discuss what properties of resonances of the second and fourth order with compactly supported coefficients are common and which are specific.

- Common properties:

- 1) The determinants $D(k)$ and $d(k)$ are exponentially type functions in terms of the variable k (not λ) and each of them has an axis of symmetry.

- 2) The resonances have the logarithmic type asymptotics for coefficients with steps.

However, there are significant differences between the resonances for Schrödinger operators and for fourth order operators. Indeed, it is well known that the resonances for the Schrödinger operators satisfy:

- specific properties of $d(k)$

- 1) The Riemann surface for the determinant $d(\lambda^{\frac{1}{2}})$ is the Riemann surface for the function $\lambda^{\frac{1}{2}}$. Then it has (as the function of λ) two sheets and $d(k) \sim 1$ as $|\lambda| \rightarrow \infty$ on the first (physical) sheet and $d(k) \sim e^{2\gamma|\operatorname{Im} \sqrt{\lambda}|}$ on the second (non-physical) sheet. There is a finite number of eigenvalues on the first sheet and an infinite number of resonances on the second one.

- 2) The determinant $d(k)$ belongs to the Cartwright class \mathcal{C}_γ . Then the Levinson Theorem describes the distribution of resonances in the large disc.

- 3) The number of resonances \mathcal{N}_r in the disk $|\lambda|^{\frac{1}{2}} < r$ has asymptotics $\mathcal{N}_r = \frac{2\gamma}{\pi}r(1 + o(1))$ as $r \rightarrow \infty$.

- 4) In order to obtain an analytic extension of the determinant from the first sheet onto the second one uses one identity (2.11).

- specific properties of the determinant D for the fourth order operator:

- 1) The Riemann surface for the determinant $D(\lambda^{\frac{1}{4}})$ is the Riemann surface for the function $\lambda^{\frac{1}{4}}$. Thus one has four sheets: $D \sim 1$ at $|\lambda| \rightarrow \infty$ on the first sheet, $D \sim e^{2\gamma|\lambda|^{\frac{1}{4}}}$ on the second and fourth sheets and $D \sim e^{2\sqrt{2}\gamma|\lambda|^{\frac{1}{4}}}$ on the third sheet. There is a finite number of eigenvalues on the first (physical) sheet and an infinite number of resonances on the other (non-physical) sheets. The number of resonances in the large disc on the third sheet is, roughly speaking, in two times more than on the second (or fourth) sheet.

- 2) The determinant D is not in the Cartwright class.

- 3) The number of resonances \mathcal{N}_r in the disk $|\lambda|^{\frac{1}{4}} < r$ has asymptotics $\mathcal{N}_r = \frac{4\gamma}{\pi}r(1 + o(1))$ as $r \rightarrow \infty$.

- 4) In order to obtain an analytic extension of the determinant from the first sheet onto the other sheets we need to use two identities (3.22), (3.23).

Acknowledgments. The study was supported by the RFBR grant No 16-01-00087.

REFERENCES

- [AP08] Aktosun, T., Papanicolaou, V. G. Time evolution of the scattering data for a fourth-order linear differential operator. *Inverse Problems*, 24(5) (2008), 055013.
- [BK14] Badanin, A., Korotyaev, E. Sharp eigenvalue asymptotics for fourth order operators on the circle. *J. Math. Anal. Appl.* 417 (2014), 804–818.
- [BK15] Badanin, A., Korotyaev, E. Inverse problems and sharp eigenvalue asymptotics for Euler-Bernoulli operators. *Inverse Problems* 31 (2015), 055004.

- [BK16] Badanin, A., Korotyaev, E. Determinant and fundamental solutions for 4-th order operators. Preprint, 2016.
- [B85] Beals, R. The inverse problem for ordinary differential operators on the line. *American Journal of Mathematics*, 107(2) (1985), 281–366.
- [BDT88] Beals, R., Deift, P., Tomei, C. Direct and inverse scattering on the line, *Mathematical surveys and monograph series*, No. 28, AMS, Providence, 1988.
- [B68] Butler, J. B. On the inverse problem for differential operators of fourth order with rational reflection coefficients. *Journal of Differential Equations*, 4(4) (1968), 573–589.
- [F63] Faddeyev, L. D. The inverse problem in the quantum theory of scattering. *Journal of Mathematical Physics*, 4(1) (1963), 72–104.
- [F97] Froese, R. Asymptotic distribution of resonances in one dimension. *Journal of differential equations*, 137 (1997), 251–272.
- [Ga] Gamow, G. Zur Quantentheorie des Atomkernes, *Zeitschrift für Physik*, 51(1928), no 3-4, 204–212.
- [H99] Hitrik, M. Bounds on scattering poles in one dimension, *Commun. Math. Phys.* 208(1999), 381–411.
- [HLO06] Hoppe, J., Laptev, A., Östensson, J. Solitons and the removal of eigenvalues for fourth-order differential operators, *Int. Math. Res. Not.*, (2006), 14 pp.
- [IK12] Isozaki, H., Korotyaev, E. Inverse problems, trace formulae for discrete Schrödinger operators. *Annales Henri Poincaré*, 13 (2012), 751–788.
- [I88] Iwasaki, K. Scattering Theory for 4-th Order Differential Operators, I. *Japanese journal of mathematics*. New series, 14(1) (1988), 1–57.
- [I88x] Iwasaki, K. Scattering theory for 4-th order differential operator, II. *Japanese journal of mathematics*. New series, 14(1) (1988), 59–96.
- [Ko88] Koosis, P. The logarithmic integral I, Cambridge Univ. Press, Cambridge, London, New York 1988.
- [K03] Korotyaev, E. Characterization of the spectrum of Schrödinger operators with periodic distributions. *Int. Math. Res. Not.*, 37 (2003), 2019–2031.
- [K04] Korotyaev, E. Inverse resonance scattering on the half line. *Asymptot. Anal.* 37 (2004), no. 3-4, 215–226.
- [K04x] Korotyaev, E. Stability for inverse resonance problem. *International Mathematics Research Notices* 2004, no. 73 (2004), 3927–3936.
- [K05] Korotyaev, E. Inverse resonance scattering on the real line. *Inverse Problems* 21.1 (2005), 325–341.
- [K12] Korotyaev, E. Estimates of 1D resonances in terms of potentials, to be published in *Journal d'Analyse Mathématique*. arXiv:1209.2109
- [K16] Korotyaev, E. Resonances of third order differential operators. (2016) preprint arXiv:1605.01842 .
- [LSSO06] Laptev, A., Shterenberg, R., Sukhanov, V., Östensson, J. Reflectionless potentials for an ordinary differential operator of order four. *Inverse problems*, 22(1) (2006), 135–153.
- [L71] Levin, B. Entire functions. MSU, Moscow, 1971.
- [L69] Leissa A. W. Vibration of plates. Ohio State Univ Columbus, 1969.
- [MSW10] Marletta, M.; Shterenberg, R.; Weikard, R., On the Inverse Resonance Problem for Schrödinger Operators, *Commun. Math. Phys.*, 295(2010), 465–484.
- [MM12] Mikhailets, V. A., Molyboga, V. On the spectrum of singular perturbations of operators on the circle. *Mathematical Notes*, 91(3) (2012), 588–591.
- [N07] Nedelec, N. Asymptotics of resonances for a Schrödinger operator with vector values. *Journal of Functional Analysis*, 244 (2007), 387–398.
- [RS72] Reed, M., Simon, B. *Methods of modern mathematical physics. Vol. 1. Functional analysis*, Academic Press, New York, 1972.
- [RS75] Reed, M., Simon, B. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press, New York, 1975.
- [RS79] Reed, M., Simon, B. *Methods of Modern Mathematical Physics, Vol. III: Scattering Theory*, Academic Press, New York, 1979.
- [RS78] Reed, M., Simon, B. *Methods of Modern Mathematical Physics, Vol. IV: Analysis of Operators*, Academic Press, New York, 1978.
- [S00] Simon, B. Resonances in one dimension and Fredholm determinants, *J. Funct. Anal.* 178 (2000), no. 2, 396–420.

- [S05] Simon, B. Trace ideals and their applications. Second edition. Mathematical Surveys and Monographs, 120. AMS, Providence, RI, 2005.
- [SZ91] Sjöstrand, J.; Zworski, M. Complex scaling and the distribution of scattering poles. J. Amer. Math. Soc. 4 (1991), no. 4, 729–769.
- [TW59] Timoshenko, S. P., Woinowsky-Krieger, S. Theory of plates and shells. McGraw-hill, 1959.
- [Z87] Zworski, M. Distribution of poles for scattering on the real line, J. Funct. Anal. 73(1987), 277–296.
- [Z89] Zworski, M. Sharp polynomial bounds on the number of scattering poles of radial potentials. J. Funct. Anal. 82 (1989), no. 2, 370–403.
- [Z02] Zworski, M. SIAM, J. Math. Analysis, "A remark on isopolar potentials" 82(6), 2002, 1823-1826.

SAINT-PETERSBURG STATE UNIVERSITY, UNIVERSITetskAYA NAB. 7/9, ST. PETERSBURG, 199034
RUSSIA, AN.BADANIN@GMAIL.COM, A.BADANIN@SPBU.RU, KOROTYAEV@GMAIL.COM, E.KOROTYAEV@SPBU.RU